ON A BAYESIAN METHODOLOGY TO THE SOLUTION OF THE NAVAL ASW SCREEN PLACEMENT PROBLEM.

by

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ON A BAYESIAN METHODOLOGY TO THE SOLUTION OF THE NAVAL ASW SCREEN PLACEMENT PROBLEM

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

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The Ohio State University
1970

PREFACE

The purpose of this investigation into a particular naval problem is twofold. First, an attempt has been made to establish a theoretical structure by which current or proposed doctrines and procedures that deal with protecting ships against submarines can be evaluated or compared. The second is to demonstrate the feasibility of using Bayesian Decision Theory to investigate complex naval problems.

While a scholarly reader may be unfamiliar with the terminology associated with the naval problem: line efficiency, BT drop, screen commander, so will the Navy reader precariously approach the standard phrases used in Bayesian Decision Theory: prior distribution, loss function, expected value of sample information, and so forth. Thus, for either type of reader to fully evaluate this paper he must refrain from just a cursory glance or reading of the summary of the results and, attempt to study the paper in its entirety. Though the mathematics used herein require a knowledge of calculus and probability theory, meaningful interpretations have been included so that the results can be understood by all. Any agreement by a reader with the results of this paper I hope will be due to the logical arguments which are used in the investigation and not due to any proof by confusion or intimidation.

I owe much to many who made possible my being in the position to take advantage of the educational opportunity made available by the

United States Navy. There were many teachers, instructors, and professors who contributed to my educational development. To each, my heartfelt thanks. In particular, I gratefully thank my adviser, Professor William T. Morris, to whom I am extremely indebted for his patient guidance, academic direction, and meaningful encouragement. Finally, as willing as I am to share any good which might result from this dissertation, I am unwilling to share any inadequacies, shortcomings, or criticisms--these are mine, mine alone.

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LIST OF SYMBOLS

$p(s)$	probability of submarine hitting
C_p	contour of isoprobability of submarine hitting
ASW	antisubmarine warfare
s	spacing between adjacent screening units
k	slope of linear isoprobability function
r	distance of the screen from the main body of ships
LE	line efficiency function of a screen
U	undetected penetration function
D_p	payoff decision matrix
BT	bathythermograph
s_o	optimal spacing
x	lateral range
$p(x)$	conditional probability of detection of a submarine with lateral range x
L	loss or increase in probability of submarine hitting
$g(s_o)$	gain function
PIP	Probabilistic Information Processing System
$PR(p\|s)$	prior distribution for submarine hitting for a given screen
$PR(q\|s)$	prior distribution of a submarine's undetected penetration for a given screen
$PR(s_o)$	prior distribution of s_o
m_{pr}	prior mean of s_o
V_{pr}	prior variance of s_o

I	amount of information
m_s	sample value of s
$LK(m_s \mid s_o)$	likelihood of a sample value of s_o
V	variance of sample value of s_o
c	assurance ratio of the likelihood variance of m_s to the prior variance of s_o
$PO(s_o \mid m_s)$	posterior distribution of s_o given a sample value m_s
m_{po}	posterior mean of s_o
V_{po}	posterior variance of s_o
$PR(m_{po})$	prior distribution of the posterior mean of s_o
$E(m_{po})$	prior expected value of the posterior mean of s_o
$V(m_{po})$	prior variance of the posterior mean of s_o
s_1	present spacing
s_2	spacing during BT drop
$U(m_{po}, s_1)$	a particular undetected penetration function evaluated at s_1
$U(m_{po})$	U evaluated at m_{po}
PREP	prior expected payoff
$\tilde{s}$	choice of spacing for $E(p \mid s) = E(q \mid s)$
$E[g(s_o)]$	expected value of gain function
$\approx$ over s	Choice of spacing to minimize $E[g(s_o)]$
POEP	posterior expected payoff
$L \mid SI$	loss given sample information
VSI	value of sample information
EVSI	expected value of sample information

I_1	integral in expression for EVSI
I_2	integral in expression for EVSI
ENGSI	Expected Net Gain from Sample Information
$L_{RH}(s_1)$	Right Hand Linear Normal Loss Quantity
$L_{LH}(s_1)$	Left Hand Linear Normal Loss Quantity
$SD(m_{po})$	prior Standard Deviation of posterior mean of s_o
$\bar{U}(m_{po})$	loss for a given m_{po} and s_1
α	parameter for linear approximation
U_L	linear approximation for U
ϕ	least probability of undetected penetration for revised strategy
s_ϕ	optimal spacing for revised strategy

CHAPTER I

INTRODUCTION

In a decision-making situation the naval commander has always been called on to know the sources of information and their reliability, to weigh the significance or importance of the various inputs, and to subjectively integrate these inputs within himself in order to arrive at a decision. This paper deals with a particular naval commander and a particular decision. But first, consider the following quotations because it is from a seed of thought, fertilized by the background represented by these remarks, that this paper grew:

> Far too many words, both classified and unclassified, are published on protecting carrier strike systems, without accompanying objectivity and understanding as to what the actual pay-off is. (10-94)

> In a problem as complex, for example, as antisubmarine warfare, judgment factors rather than applied mathematics will frequently predominate. At present, ASW study is characterized by broad areas of uncertainty and, therefore, by many factually unsupportable judgments, either implicit or explicit. (18-60)

> Ever since the end of World War II, naval commanders have sought some way to access antisubmarine warfare systematically. (2-72)

> Naval experience and judgment can and should be applied from the beginning of the analytical process for realistic, practical results. (23-199)

> One must be willing to inject judgments in usable form into the analytical process; be knowledgable enough to understand its strengths and weaknesses; and be courageous enough to accept the results as the best available basis for decision making. This turns out to be a very large order, indeed. (1-48)

Even though the naval screen placement problem is only one of a number of complex ASW problems, we can see that a systematic solution approach is called for which incorporates judgment with an objective or quantifiable payoff. Such an attempt is made in this paper.

The nature of our approach will be to investigate the screen placement problem in three phases: deterministic, probabilistic, and informational. The deterministic phase includes the first six chapters. Chapter II and Chapter III can be combined to represent the current naval treatment of our problem and is mainly derived from References (17) and (22). Chapter IV provides an identification of the decision problem which faces the naval commander. Chapter V, which in part summarizes Koopman's kinematic search theory, is used to develop in Chapter VI a deterministic payoff structure. Even though there has been a great deal of interest in search theory as summarized in References (4) and (8), there has been no attempt to elaborate on the "optimal screening rule" put forth in Reference (17) in 1946. This is the first attempt known to the author.

The probabilistic phase of our study is developed in Chapter VII and Chapter VIII. Wherein Chapter VII summarizes the proposal and extensive studies made by Edwards et al in using probabilistic information process systems to model command and control decision situations, this paper represents the first attempt,again known to the author, to incorporate such a system with a payoff in order to investigate the ASW problem with which we are concerned. Chapter VIII includes the development of the probability distributions that is required to

conduct the economic analysis. The informational phase is conducted in Chapter IX in which the economic benefits will be shown if information is established. Chapter X interprets the results of our investigation. A numerical example is presented as Chapter XI. The sensitivity of some of the important assumptions upon which the analysis is based is examined in Chapter XII. In Chapter XIII the results of the analysis are generally discussed, recommendations for implementation made, and future research suggested. A brief summary makes up Chapter XIV.

The scope or ultimate intention of the paper is for it not only to represent a theoretical analysis with which doctrine can be evaluated, but also to imply the use of Bayesian Decision theory in solving similar complex ASW decision problems. The degree to which this intention is accomplished is, of course, judgmental. However, it may be well to note early in the undertaking of the analysis that the major limitation which must be overcome before this theoretical analysis can become operational or put into practice in the fleet may very well be lack of the theoretical bases from which certain deterministic procedures are developed. In particular in our case, the "optional screening rule" must be understood before any judgmental implementation can be investigated or put into operation.

In order for the reader who lacks a naval background to obtain an appreciation of that portion of the complex problem of protecting surface ships in a submarine environment that is analyzed in this

paper, we present a brief discussion of an example of the overall anti-submarine warfare operation as it may pertain to an aircraft carrier and a group of destroyers.

When these ships leave a port it is usually the destroyers that get underway first so that they may sweep a channel through which the carrier may pass as it proceeds to the open seas or deeper water. As the destroyers sweep the channel they are using their acoustic sensing devices to attempt to locate any possible submarine concealed in the water. As the carrier exits the swept channel, the destroyers group to form a previously ordered screen in order to continue to search for lurking submarines. The geometrical arrangement of the screening destroyers, relative to the carrier, will depend on the number of destroyers available and the relative speed between the carrier and anticipated submarine. Another relative speed, that between the carrier and the submarines torpedo; is used to estimate the potential capabilities of the submarine's weapons. The distance from the carrier at which the destroyer will be formed into a screen is a function of this latter estimate and the presumed performance of the sonars of the destroyers which in turn depend on water conditions. Because these water conditions can change, there is a need for determining when these changes take place. Presently this situation is coped with in a periodic and deterministic manner - basically, a look every four hours. However, because a reduction of the detection capability of the destroyer screen occurs when a ship is detached to investigate a possible

change in water conditions, the question of when to investigate is extremely important.

The main goal of the destroyer screen is to prevent the submarine from hitting the carrier with torpedoes. This may be accomplished at times in either a defensive or offensive manner. As the screen sweeps the water through which the carrier transverses, a defensive role is being played by the destroyers. Assuming the commander of an attacking submarine can evaluate the options of either shooting from outside the screen or first attempting to penetrate the screen and then shooting, the spacing between destroyer, or, in other words, the distance from the carrier is chosen to minimize the chance of a submarine accomplishing its mission. However, upon any detection or attack by the submarine some of the destroyers may go on the offensive and attack the submarine, while the carrier with the remaining destroyers proceeds away from the direction of submarine threat. This offensive/defensive characteristic of the destroyer screen is maintained until the danger of submarine attack is no longer present, which in turn might only occur when the ships are in a protected port.

Thus, as previously stated, the objective of this investigation is to use Bayesian Decision Theory to analyze for a given estimate of the threat of an anticipated submarine, the defensive role of the screen in the ASW problem. That is, an attempt to take into account the experience and judgment of naval screen commanders in order to determine both the distance from the carrier at which the screen should be formed and

also how often to obtain additional information about the water conditions is the main goal of this theoretical analysis. Though the tactical problems of attacking the submarine are important, they are not considered herein. It is felt that a separate look at the defensive problem may be acceptable because some solution to this problem is usually required prior to the need for solving the offensive or tactical problems.

CHAPTER II

THE NAVAL ANTISUBMARINE WARFARE SCREENING PROBLEM

The Screening Problem

The naval antisubmarine screening problem can be described as that problem associated with a naval operation in a submarine warfare environment which considers the effective utilization of screening units about a formation of ships in order to protect or defend this formation against a submarine attack. The formation or main body can consist of several ships such as a task force, replenishment group, or merchant convoy, or it can be composed of only a single ship such as an aircraft carrier or fleet oiler. The screening units are more maneuverable, armed craft such as destroyers and are employed to detect and possibly destroy submarines at as great a range from the formation as feasible.

Detection

The unique characteristic of a submarine from which all its other potential qualities are derived, is simply its ability to hide in the sea. Thus, a submarine must be detected before any reasonable attempt can be made to destroy it. In general, two basic requirements must be met if any detection is to occur. The first requirement is a certain set of physical conditions, such as light being reflected from an object, that must be satisfied, and which, if fulfilled, may allow detection. Second, there must be an awareness that the physical conditions have been fulfilled. How many of us have looked directly at an object

in plain view and not detected its presence? It seems reasonable to accept detection as an event, the occurrence of which can be described by a conditional probability function, which yields the probability of detection, given the target is present. This probability will simply be referred to as the probability of detection and the condition of the target's presence will be implied.

Sonar Detection

Detection by screening units is accomplished by using active sonar to obtain glimpses into the sea. Active sonar is analogous to a searchlight probing the sky at night. That is, sonar detection entails transmitting acoustic energy from the transducer of a screening unit through the water and then waiting to learn if some of this sound energy is reflected back to the transducer after striking some object, hopefully a submarine, if it is present. Though advances have been achieved in active sonar by lowering the frequencies of operation and increasing the size and power of the transducers, the progress has been modest. Detection ranges are limited due to transmission losses in the sea. Even though more powerful transducers will result in some increase in maximum detection range, there exists a diminishing marginal return that approaches a finite limit because the acoustic energy is dissipated over more than one dimension. At the present time, and until some technological breakthrough is made, this diminishing return effect, when combined with the size and space constraints that exist when constructing a highly maneuverable screening unit, requires more

than ever the most efficient use of screening units in order to obtain as great a detection capability as possible.

Screen Commander

The screen commander is the individual who is charged with the responsibility of effectively deploying the screening units. Each possible screen that the screen commander can order represents an option for him to consider when attempting to efficiently utilize his screening units. The numerous options are systematically processed by viewing the screen commander's choice of what screen to order as a two-stage sequential process. First, he must decide the type of screen, that is, a particular geometric pattern about the main body in which to form the screening units. Then, given this first decision, he must decide how far from the main body to place the screen.

Because the type of screen is usually based on information which cannot be updated by the screen commander through any direct efforts on his part to obtain more pertinent data, we will consider the type of screen as a response to an estimate of the threat that menaces the main body. The size and speed of the main body and estimates of the speed of the attacking submarine and its weapons capabilities are some factors which the screen commander must consider when choosing the type of screen or estimating the threat.

Contours of Isoprobability

A measure of how well a screen is protecting the main body, which seems to suggest itself, is the probability that a submarine firing a torpedo from just outside the screen will hit the main body with it. Koopman (17-129) has shown how the size of the main body and estimates of the capabilities of the weapons belonging to the submarine can be combined to yield the probability of hitting a single ship of the main body with a single torpedo. This probability will be referred to as the probability of submarine hitting and will be denoted by p(s). Because the relative position of the submarine with respect to the main body is a major factor in determining the probability of submarine hitting, submarines at different relative positions will have different probabilities of submarine hitting. Following the procedure demonstrated on a nautical chart when locations in the ocean, which have the same depth, are connected on a chart with a line to form a contour of isodepth, we connect the relative positions which have the same probability of submarine hitting in order to form contours of isoprobability. As we will see later, the notion of a contour of isoprobability will be convenient when we further investigate the screen commander's options. Let us now turn our attention to the development of these contours, denoted Cp, of equal probability of submarine hitting which we realize now are based partly on from where the submarine is firing. Because the development will vary with the type of main body the screen commander is attempting to protect, the procedure

of calculating these contours will be demonstrated below for a formation composed of a single ship. The following discussion is primarily from references (17) and (22).

Consider Figure 1 which illustrates a submarine firing from point F at a ship of length L, in relative space with respect to the ship's movement.

The angle ω depicts the angular limits of the relative tracks of the torpedoes fired from F that will hit the ship. Angle ω is a function of the relative range R_f, bearing θ of the submarine from the ship, and ship length L. To show this, first notice that using the approximation for small angles which permits the angle in radians to be expressed by the ratio of its chord and radius, here A and R_f respectively, ω can be expressed as follows

$$\omega = \frac{A}{R_f} \text{ in radians.} \qquad (2\text{-}1)$$

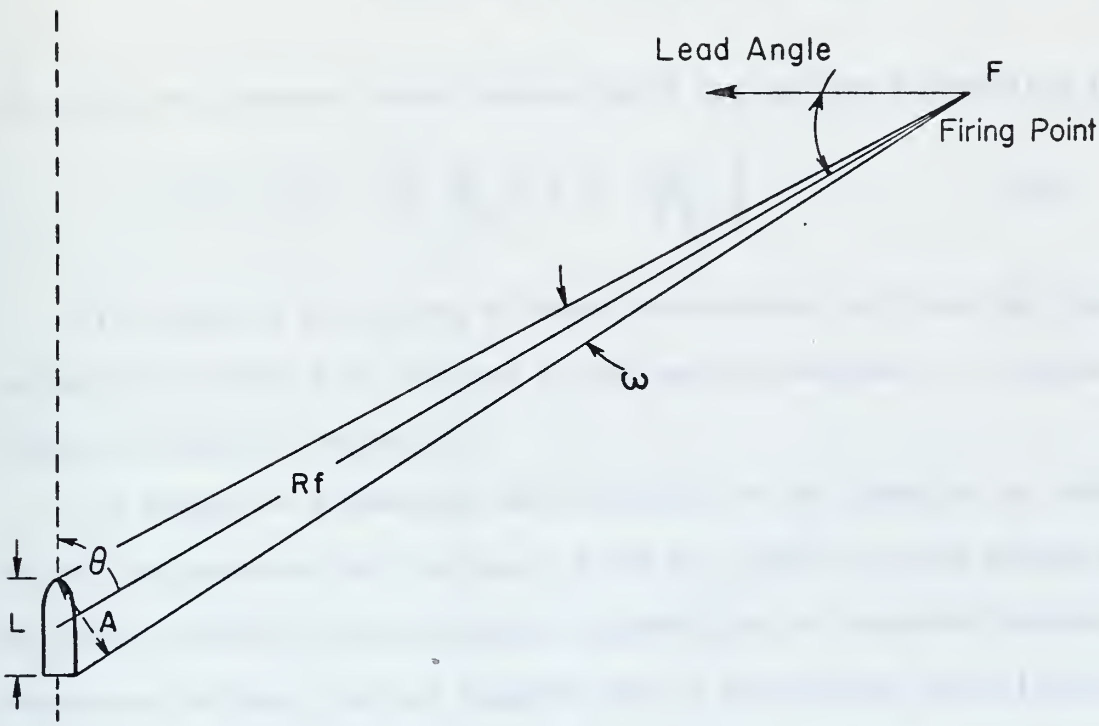
If R_f is much greater than L, then A is approximately equal to the product of L and sine of relative bearing θ. Substituting this approximation into Equation 2-1 yields

$$\omega = \frac{L \sin \theta}{R_f} \text{ in radians} \qquad (2\text{-}2)$$

or,

$$\omega = \frac{180}{\pi} \frac{L \sin \theta}{R_f} \text{ in degrees} \qquad (2\text{-}3)$$

Let us treat the firing error, denoted by y, as a random variable and assume its distribution is normal with zero mean and variance σ_θ^2



Torpedo Firing At A Single Ship

FIGURE I

which can be shown to be a function of relative bearing (14-131). Then, the probability, $P(R_f, \theta)$, of a submarine hitting from a relative range R_f at bearing θ is as follows:

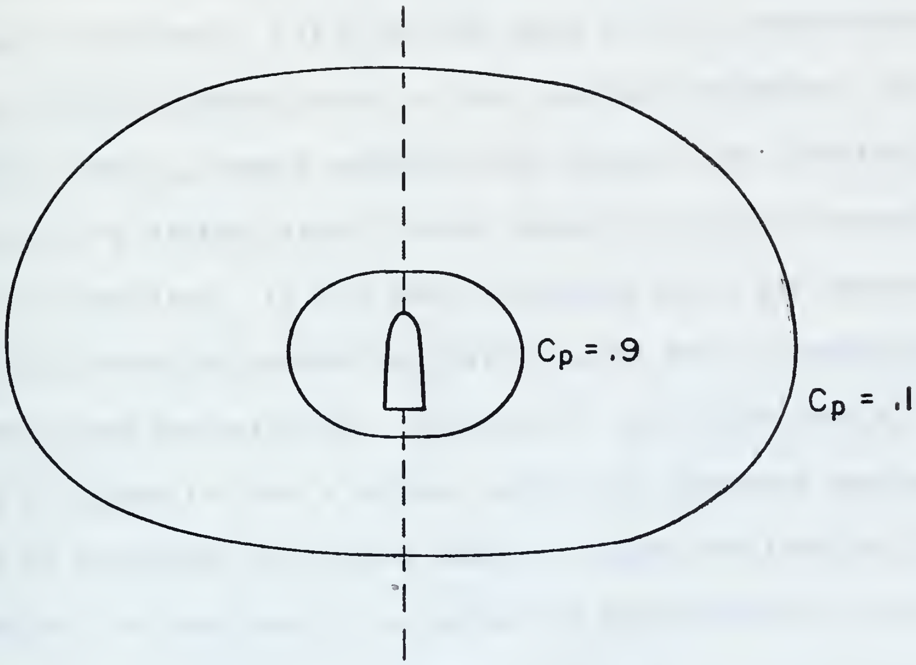
$$P(R_f, \theta) = P\left[-\frac{\omega}{2} \leq y \leq \frac{\omega}{2}\right] \qquad (2\text{-}4)$$

or, using the standard normal tables for Z the desired probability is

$$P(R_f, \theta) = P\left[-\frac{\omega}{2\sigma_\theta} \leq Z \leq \frac{\omega}{2\sigma_\theta}\right] \qquad (2\text{-}5)$$

The locus of all points of equal probability will form the iso-probability contours of interest to the screen commander. A typical contour is shown in Figure 2.

It should be emphasized that estimates of the speed of the submarine's torpedo and the variance of the hit capability are required in order to develop the contours of probability of submarine hitting. The nature of these factors suggests that a pre-sailing intelligence report may establish these estimates and indeed does not permit the screen commander an opportunity to sample for current data. Thus, we will consider the establishment of these contours as part of the estimates of the threat.



Typical Contours of Isoprobability

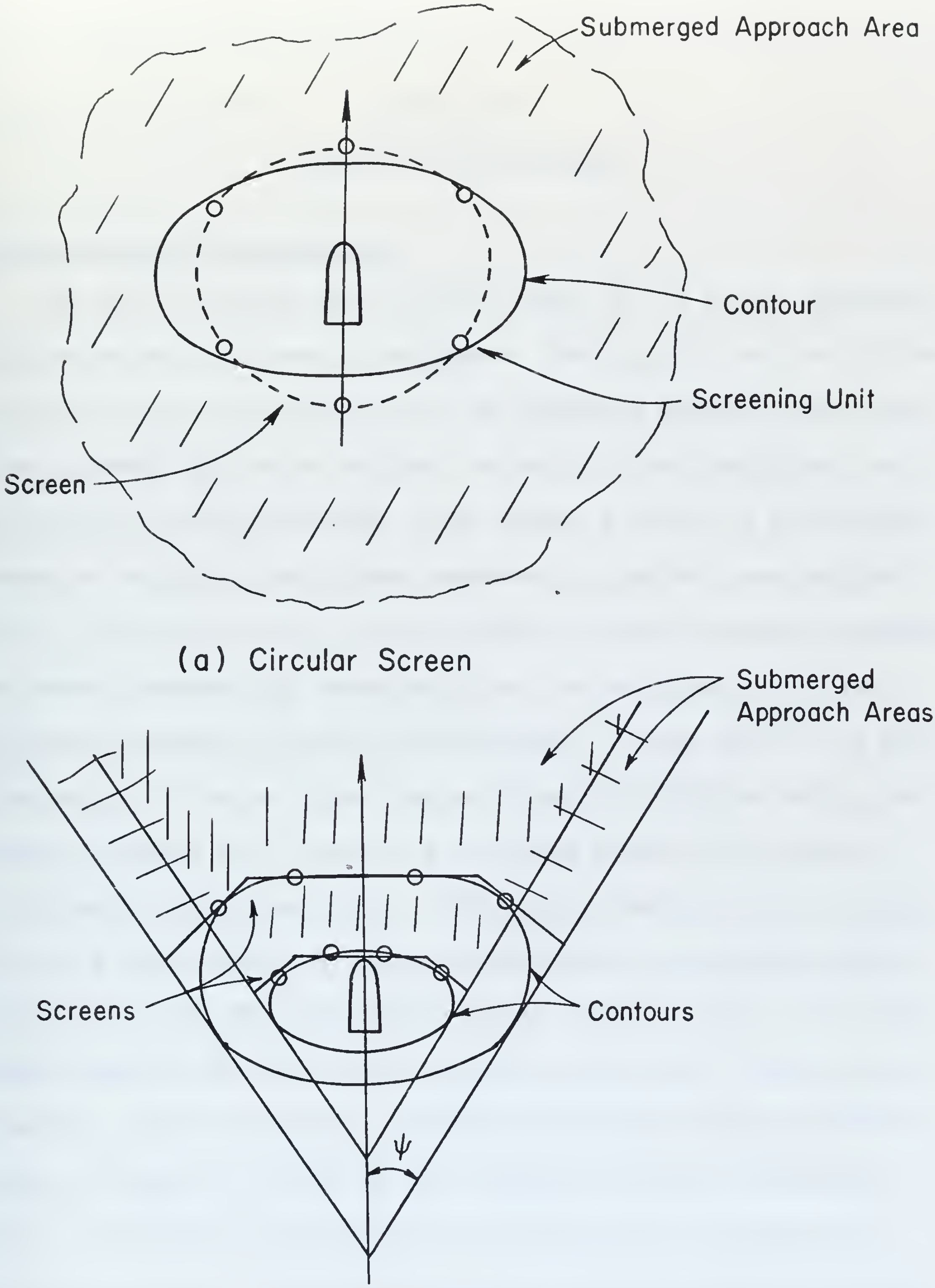
FIGURE 2

Types of Screens

We will now investigate two general types of screens which the screen commander may order. In general, the type of screen ordered depends on the type of threat against the main body. In an ASW situation, relative speed between the target and the submarine is of primary importance. Let u_t be the speed of the formation and u_s be the estimated approach speed of the submerged submarine. If u_s is greater than u_t, then a submarine may approach the formation from any direction to station itself on any desired contour of isoprobability given enough time. In this case, screening units are deployed in a circular screen to protect against a threat that is present 360 degrees around the main body. However, if u_s is less than u_t, the direction of threat is from a region, called the submerged approach region, that is determined by drawing targets, called the limiting lines of approach, to both sides of a contour of isoprobability at an angle Ψ, where

$$\Psi = \sin^{-1}\left(\frac{u_s}{u_t}\right) \text{ in degrees,} \qquad (2\text{-}6)$$

measured from the formation's course. The ASW screen that is used in this case is called a bentline screen. This screen can be constructed by drawing a line over a contour such that each end of the line is terminated perpendicularly to one of the limiting lines of approach. Notice that as the value represented by the contour decreases, the greater will be the length of the line drawn to construct the corresponding bentline screen. These two types of ASW screens are illustrated in Figure 3.



(a) Circular Screen

(b) Bentline Screens

Typical ASW Screens

FIGURE 3

CHAPTER III

PLACEMENT OF THE SCREEN

Spacing Between Screening Units

We will now assume that the first stage of the screen commander decision sequence has been accomplished. That is, the contours of iso-probability have been estimated and the submerged approach zone calculated or simply the threat has been estimated. Even though the type of screen has been established, there remains a number of alternative courses of action for the screen commander to consider when deciding where to place the screen. A controllable decision variable with which the screen commander may conveniently work is the spacing, denoted S, or distance between adjacent screening units. In the case of the circular screen, if we use equal spacing between all adjacent units, any change in spacing will result in a different screen with respect to its distance from the main body. This can be understood if we realize that for a small spacing between screening units, the screen will be closer to the main body than when a larger spacing is used. The same result holds for the case of the bentline screen also. Implicitly, the assumption, that the number of screening units which the screen commander may deploy is fixed, has been made and will hold throughout. Hence, a different spacing between screening units corresponds to a different screen or a different alternative course of action for the screen commander.

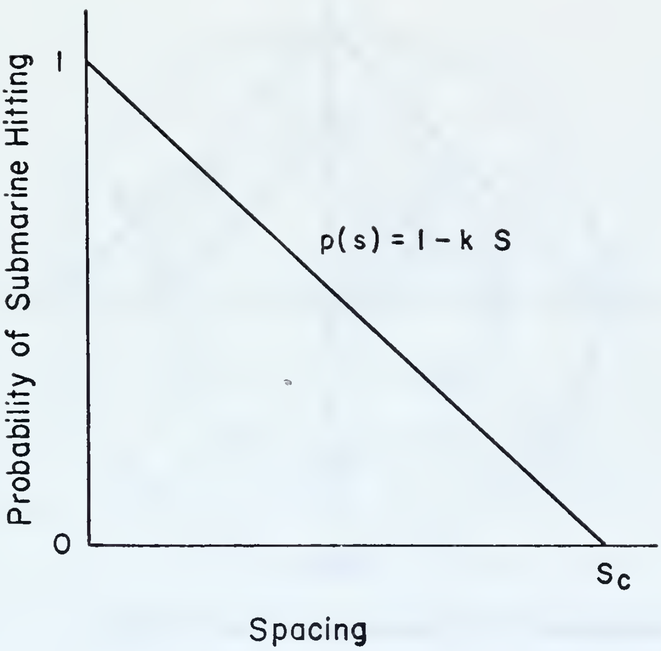
Contours Versus Spacing

Let us notice in Figure 3, that as the spacing changes, the screen tends to coincide with a different contour of isoprobability of submarine hitting. For a very close spacing the corresponding contour will have a value approximately one and for some large spacing, say S_c, the value of the corresponding contour will be almost zero. As we will see, a linear function adequately describes this relationship between a change in spacing and the corresponding change in spacing as shown in Figure 4. The reasonableness of the linear function of contour versus spacing can be seen, for example, in the case of the circular screen, because the length of the contour with which a screen roughly coincides, is approximately the circumference of a circle. The corresponding spacing between screening units is proportional to twice the sine of a fixed angle as shown in Figure 5. Thus, as the value of the contour decreases, the radius of the circle becomes larger, which results in an increase in spacing. This direct proportionality is illustrated in Figure 5. We can summarize this discussion by writing the following equation:

$$p(s) = 1 - ks \tag{3-1}$$

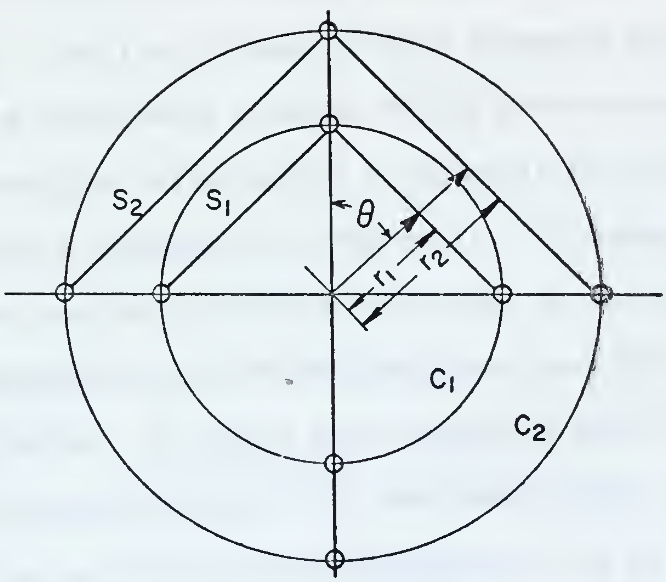
where k is the slope of the linear function.

Let us pause here to notice that if the number of screening units with which the screen commander could deploy in the screen were to become smaller, then the same spacing between units would correspond to a larger contour than before the decrease in number of units.



Probability of Submarine Hitting Versus Spacing

FIGURE 4



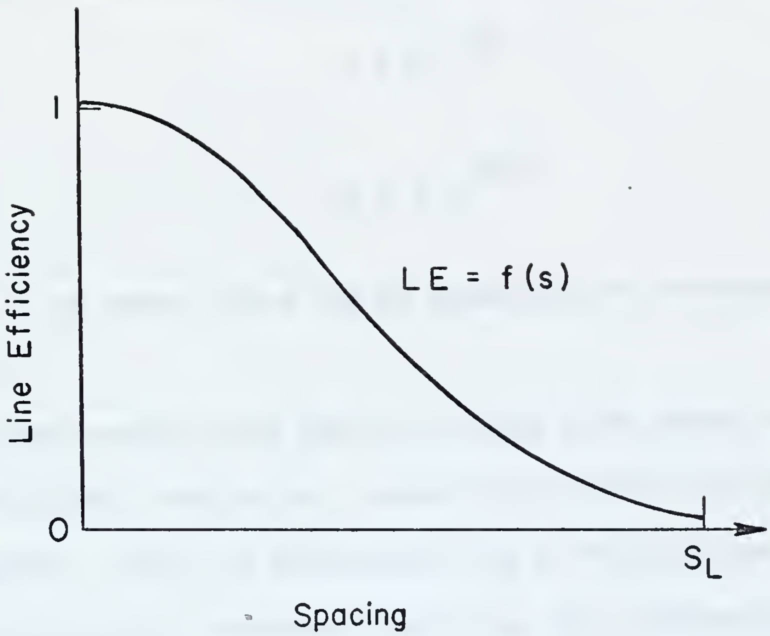
Contours with Corresponding Spacings

FIGURE 5

Or, in other words, the revised value of the slope in Equation 2-1 would be smaller than before. Recall that we have assumed the threat has been estimated; thus, because the contours are determined as part of the estimated threat, we will consider the linear function of probability of submarine hitting versus spacing as known and fixed.

Line Efficiency

Corresponding to each of the screen commander's alternative courses of action, there exists a measure called the line efficiency, denoted LE. The line efficiency of a screen is the probability that a submarine penetrating a screen with a given spacing is detected. That is, the line efficiency of a screen is the probability of detection discussed previously in Chapter II. Of course, the value corresponding to one minus the line efficiency of the screen is the probability a submarine penetrates this screen undetected, i.e., a successful penetration. It should seem reasonable that if the spacing between screening units was very, very small, then the line efficiency for this screen would be approximately one and as the spacing increases the line efficiency decreases until, for some large value of spacing, say S_L, the line efficiency becomes almost zero. A typical curve, called the line efficiency function, that shows the change in line efficiency of a screen versus a change in spacing, is portrayed in Figure 6. We will consider the line efficiency function as a monotonic decreasing function of spacing such that for zero spacing, the line efficiency is equal to one and for some large spacing, it is equal to zero.



Line Efficiency Versus Spacing

FIGURE 6

In order to develop the economic analysis so that we may obtain insight as to when sample information should be obtained, we may need to assume a specific form for the line efficiency function. Two functional forms, that satisfy the monotonic decreasing assumptions, which seem to suggest themselves as representatives of families of curves are

$$L\,E = e^{-\lambda S} \tag{3-2}$$

and

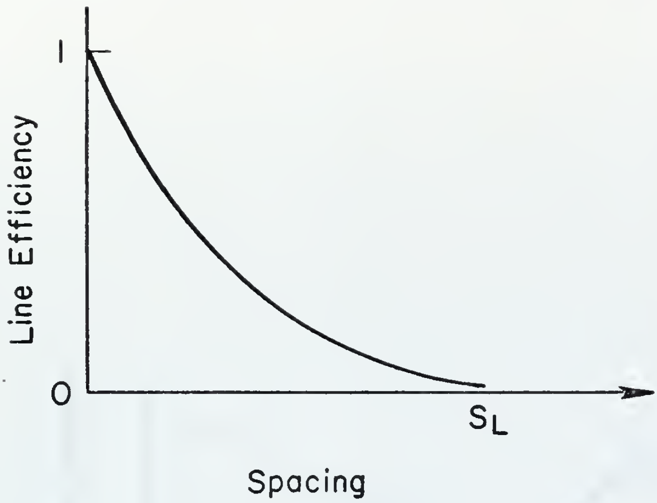
$$L\,E = e^{-\lambda S^2} \tag{3-3}$$

The generality of these curves can be visualized by examining Figures 7(a) and 7(b).

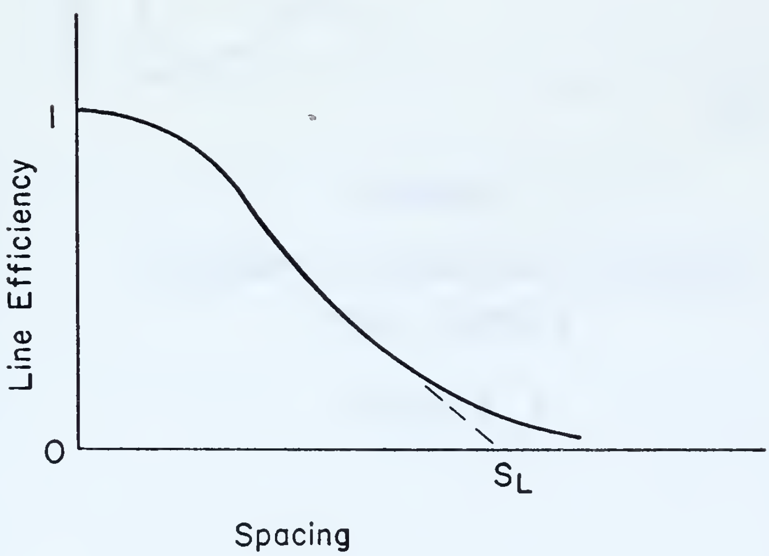
We might mention here that the second curve seems to model the line efficiency function in a manner that would be agreeable to a screen commander. From the functional form of this LE function, we can observe that as spacing increases from zero, that the probability of detection decreases slowly but eventually a more rapid decrease in probability of detection occurs. This seems to model the effect of the negative velocity gradients of sound in the ocean and their effect on the detection capability of the sonar equipment.

Because the probability of a successful penetration of a given screen is equal to one minus the line efficiency of the screen, we can graph the probability of undetected penetration as a function of spacing as shown in Figure 8.

For a particular screening situation the screen commander requires information about the sonar conditions of the water and the



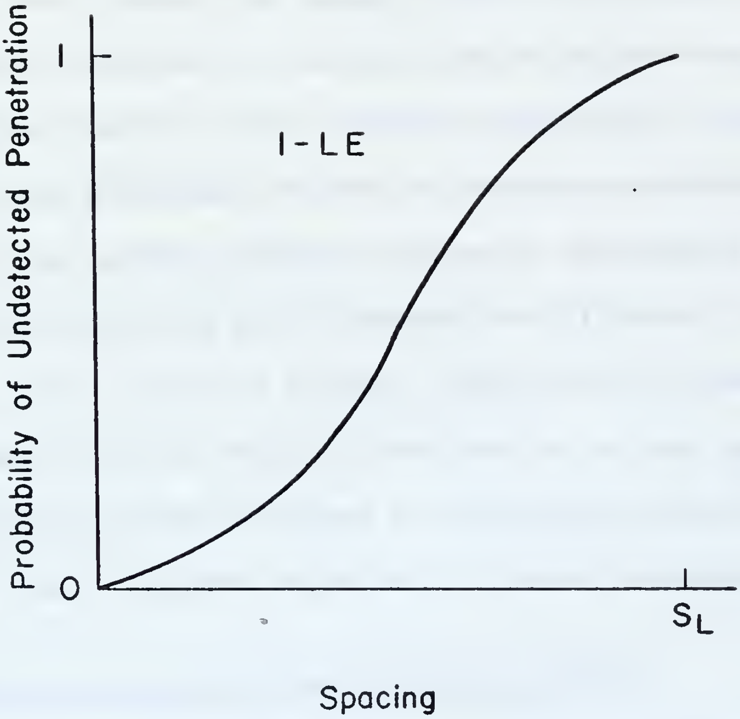
a) $LE = e^{-\lambda s}$



b) $LE = e^{-\lambda s^2}$

Candidates (2) for Line Efficiency Function

FIGURE 7



Probability of Undetected Penetration
Versus Spacing

FIGURE 8

characteristics of his sonars in order to estimate the line efficiency of a screen. How this information is combined to yield the line efficiency of his screen as a function of the spacing between screening units will be shown later. However, because this information is fallible or can even change (for example, water conditions), the screen commander will most likely be somewhat uncertain about any estimated line efficiency function. His subjective uncertainty or degree of belief in his line efficiency can best be described probabilisticly. Yet, even if the screen commander's degree of uncertainty is such that he is willing to use his estimated line efficiency function in solving the screen placement problem, what provides guidance for the screen commander when he makes his decision as to what spacing to order for his screening units? In order to investigate this question, we must first examine the main objective of a naval antisubmarine screen.

Main Objective of the Antisubmarine Warfare Screen

The main objective of an antisubmarine screen is to furnish maximum protection for the main body. Another way to view this objective is that the screen placement be utilized to minimize the submarine's chance of hitting a ship in the main body with a torpedo. To understand how this objective can be obtained requires an examination of the problem that faces the submarine commander, whose goal is to attack the main body.

The Submarine Commander's Problem

When deliberating from where to fire his torpedo at a screened formation, the submarine commander has two basic alternative courses of action. He may fire from outside the screen or he may choose to penetrate the screen and then fire from inside the screen at a closer range to the main body. When the submarine commander is faced with this decision, he must consider the two probabilities that describe these events. The probability that represents the first alternative - of firing just outside a given screen - is, of course, the probability of submarine hitting that is given by the linear function,

$$p(s) = 1-k\ s. \tag{3-4}$$

Because the second possible choice is the intersection of two events, the probability of interest can be considered as the product of the conditional probability of the submarine hitting, given undetected penetration of the screen, say g, and the probability of undetected penetration, say U. Recall we have

$$U = 1 - LE \tag{3-5}$$

For simplicity, we assume that the conditional probability of submarine hitting, given undetected penetration, is equal to one. Hence, we can equate the probability of submarine hitting, p(s), with the probability of undetected penetration, or U. Thus, if the commander of a submarine just outside the screen feels the probability of submarine hitting the main body with a torpedo fired from outside the screen is greater than the probability of penetrating the screen

undetected, he would choose to fire from outside the screen. Conversely, when successful penetration is more likely, he would attempt to penetrate and choose not to fire from outside the screen. An implicit assumption in structuring the preceding discussion is that if a submarine is detected as it attempts to penetrate the screen, then the probability of scoring a hit on the main body is zero. This is reasonable because the submarine will most likely be maneuvering to survive an attack by the screening units.

Game Theoretic Model of the Screen Placement Problem

From the examination of the submarine commander's problem, the analysis of this military conflict situation as a game of strategy in which a player's skill and intelligence should be used to determine the payoff, seems to suggest itself. This formulation of the screen placement problem is intended to structure the situation faced by the screen commander as a two-person, finite, complementary, pseudo-cooperative game.

It is two-person since there are only two opponents, the screen commander and the submarine commander. Because the submarine commander has only two options and the screen commander some finite number of possible spacings to order, the game is finite.

The game will be considered pseudo-cooperative because even though the two opponents do not directly communicate with each other, nevertheless, the submarine commander will be able to observe the screen commander's decision as to what spacing to order prior to making his own decision whether to fire or penetrate.

Because of the probabilistic nature of the payoff, consideration of the screen placement problem in the complementary sense means that whatever the screen commander calculates the probability of submarine hitting for some spacing and option by the submarine commander, the complementary event, as measured again by the screen commander, will describe the submarine commander's chance of not hitting. It follows then that the sum of the screen commander's payoff and the complement of the submarine commander's payoff determined in this manner will be one.

For such a game as being considered, a payoff matrix or array of the payoffs (probability of submarine hitting) to either player, resulting from all combinations of the players' strategies, can be constructed. If M and 2 are the total number of options available to the screen commander and the submarine commander, respectively, then a payoff matrix, D_p, can be constructed, such that the number of rows and the number of columns is equal to the number of options available to the screen commander and submarine commander, respectively. This matrix then, completely describes all possible outcomes. An element of D_p, denoted d_{ij}, represents the probability of submarine hitting if the screen commander uses the i^{th} spacing and the submarine commander chooses the j^{th} option. Thus, D_p is an M by 2 matrix with elements d_{ij}, such that $i = 1,2,\ldots, M$ and $j = 1, 2$.

Optimal Screening Rule

In solving for the optimal strategy in such a matrix game, the screen commander will apply the minimax criterion. Under this criterion the screen commander makes use of the following three presuppositions.

First, the screen commander feels that the attacking submarine commander's motives are diametrically opposed to his own. The screen commander is trying to get the main body intact across the water and the submarine commander is attempting to prevent this deed.

Next, the screen commander realizes that the submarine commander could very closely approximate the screen commander's payoff matrix and determine if the spacing used by the screening units is the optimal in the minimax sense.

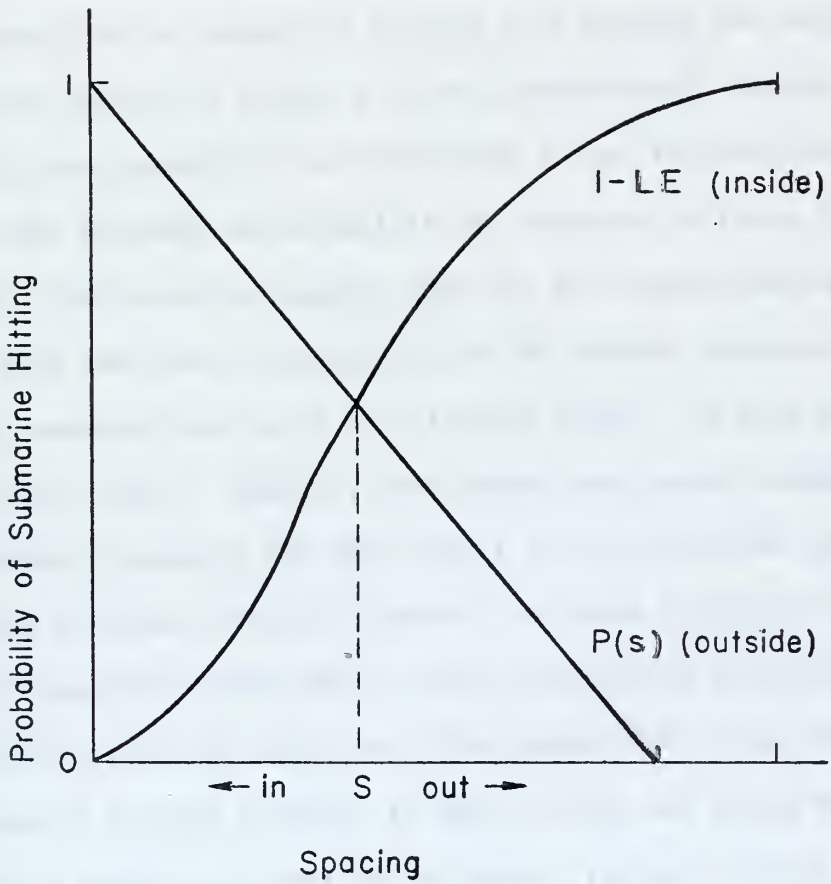
Finally, the screen commander feels that if the submarine commander knew the screen commander's choice of spacing, then the submarine commander would choose his option so as to increase the screen commander's payoff, or, in other words, obtain the larger possible probability of submarine hitting. These three presuppositions indicate that the screen commander considers the submarine commander a rational and intelligent opponent in the game theory sense.

With these factors as a basis for his criterion, the screen commander begins his selection of his optimal strategy for his screen placement by investigating the worst that could happen; i.e., the largest value in a row of the D_p matrix, for each of his possible alternatives. He then takes the alternative corresponding to the minimum of these maximum values as his optimal strategy. This is the well-known

minimax strategy of game theory. Although this criterion is usually pessimistic in nature and its use generally provides an upper bound or the worst that could result, the pseudo-cooperative nature of the game which models the screen commander's problem makes the minimax criterion a very reasonable one with which to evaluate the screen commander's options.

If we combine Figures 4 and 8 to obtain Figure 9, we will obtain a continuous representation of the game theoretic discussion that we have just presented. Because the probability of submarine hitting from outside is a linear function of spacing with a negative slope, and the probability of submarine hitting from inside increases as spacing increases, the minimax optimal choice of spacing is determined by choosing that spacing which corresponds to the intersection of the two functions. This in game theory terms is a pure strategy and results in an optimal screening rule which suggests that the spacing should be chosen that makes the probability of submarine hitting from just outside the screen equal to the probability of successful penetration, when the probability of hit from inside the screen is assumed equal to one, if a successful penetration of the screen by the submarine occurs.

If Figure 9 is examined further, it can be seen that if the screen spacing ordered by the screen commander is greater than the spacing corresponding to the intersection of the curves, then the probability of successful penetration by submarine is greater than the



Probability of Submarine Hitting Versus Spacing

FIGURE 9

probability of submarine hitting from just outside the screen. The converse also holds, that is, if the screen is moved in closer to the main body than the optimal screening rule dictates, then the probability of submarine hitting from inside is less than the corresponding probability of submarine hitting from outside the screen. Notice that with the aid of Figure 9 we can heuristically measure with respect to the probability corresponding to the intersection of the curves any increase in probability of submarine hitting for any screen option. This seems to suggest that for any screen-spacing ordered, other than that one corresponding to the optimal screening rule, the screen commander may incur some loss of sorts. We will have more to say on this later. However, even though the screen commander's minimax strategy presents the same payoff to the submarine commander regardless of which option he chooses, it seems intuitive that the submarine commander should use a mixed strategy and randomly choose between his options in this case. The reason behind the submarine commamder's use of a mixed strategy is that it will not allow the screen commander to predict his actions and hence, in some devious way, decrease the probability of submarine hitting.

Thus, we have developed an optimal screen placement rule for a deterministic, static game situation. It is deterministic in the sense that the payoffs that are entered in the decision matrix are precisely known and static in that these payoffs do not change. However, for the typical screen commander, he may not only be uncertain about the

values that enter the payoff matrix, but these values or estimates of probability of submarine hitting may change as time goes by. This suggests that we should investigate both, how the screen commander can implement his optimal screening rule when uncertainty is present, and when he should attempt to update his payoff matrix. For now, however, we shall use this game theoretic, partial structure of the screen placement problem to identify the decision problem that is faced by the screen commander.

CHAPTER IV

DECISION PROBLEM

Layer Depth

The mixing action of the wind and waves causes the temperature of the water to be constant for some depth near the surface. The depth of this layer of isothermal conditions is important because below this depth there usually exists a strong negative thermal gradient which causes the sound waves to curve downward as they travel through the water. Hence, below the layer depth the sound waves reflected from a submarine are also bent downward, lessening the chance of the reflected energy being propagated back to the transducer of the transmitting unit. It will be indicated later how the layer depth affects the line efficiency of a screen. Now we need only to realize that the submarine has continuous knowledge of the layer depth and can most likely go below this depth when penetrating a screen and thus hide more effectively than if it were above the layer depth.

Sources of Information

We have already mentioned that the screen commander uses the information about the thermal gradient of the water and the characteristics of his sonar in order to estimate the line efficiency function. The screen commander has basically two sources of information about the sonar condition of the water. The first is an atlas that shows the past average sonar conditions over a large portion of the world.

Figures 10 and 11 illustrate sample information contained in this atlas. Thus, for some previous calendar month the composite layer depth and sea surface temperature with the respective frequencies of observations is available to the screen commander. Thus, if the screen commander has a chart in the atlas that describes his present operating area, then this will provide some insight as to what to expect the current thermal gradient to be.

The current conditions are determined by dropping an instrument called a bathythermograph, BT, into the water and lowering it to a great depth while the ship proceeds at a relatively slow speed. This instrument, because it is attached to a cable, is then retrieved. A slide inside the BT, upon which is recorded the temperature of the water at various depths, gives the current information about the thermal gradient in the sea. This particular information is used subsequently to yield the line efficiency function. How this is accomplished will be indicated in the next chapter.

Because of a time lag in processing the BT data, though only a few minutes, let us realize that when the BT information is utilized, it is possible that it may no longer be applicable because the screening units are no longer at the exact location where the information was gathered. Though this is most likely not the case, it does point out that any line efficiency of a screen is subject to change as the screening units proceed through the sea. Because the line efficiency can change, the accuracy (or lack of accuracy) of the line efficiency, may cause the screen commander to ponder about the appropriateness of his currently ordered spacing between screening units.

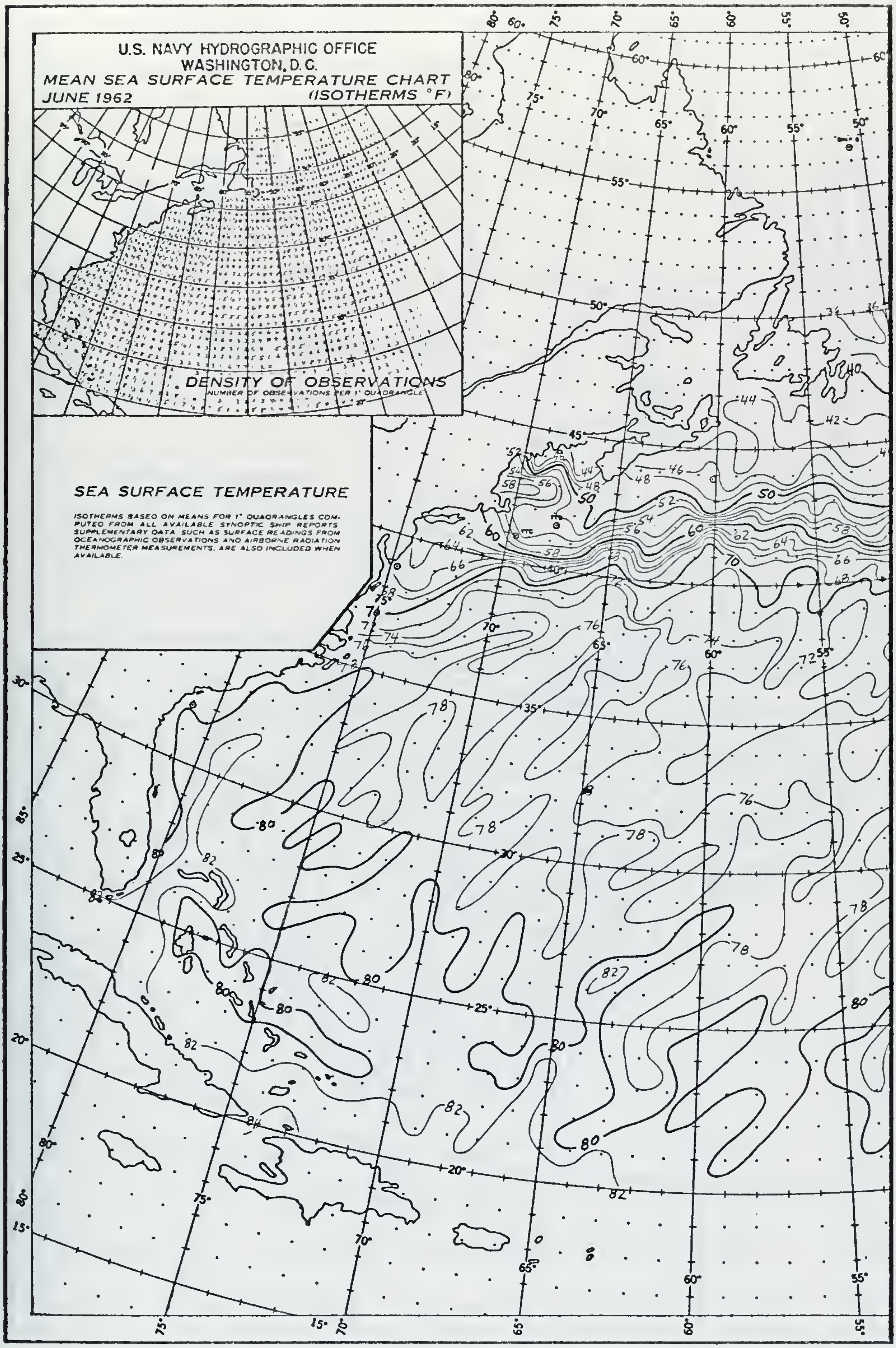

U.S. NAVY HYDROGRAPHIC OFFICE
WASHINGTON, D.C.
MEAN SEA SURFACE TEMPERATURE CHART
JUNE 1962
(ISOTHERMS °F)
DENSITY OF OBSERVATIONS
NUMBER OF OBSERVATIONS PER 1° QUADRANGLE
SEA SURFACE TEMPERATURE
ISOTHERMS BASED ON MEANS FOR 1° QUADRANGLES COMPUTED FROM ALL AVAILABLE SYNOPTIC SHIP REPORTS. SUPPLEMENTARY DATA SUCH AS SURFACE READINGS FROM OCEANOGRAPHIC OBSERVATIONS AND AIRBORNE RADIATION THERMOMETER MEASUREMENTS, ARE ALSO INCLUDED WHEN AVAILABLE.

FIGURE 10

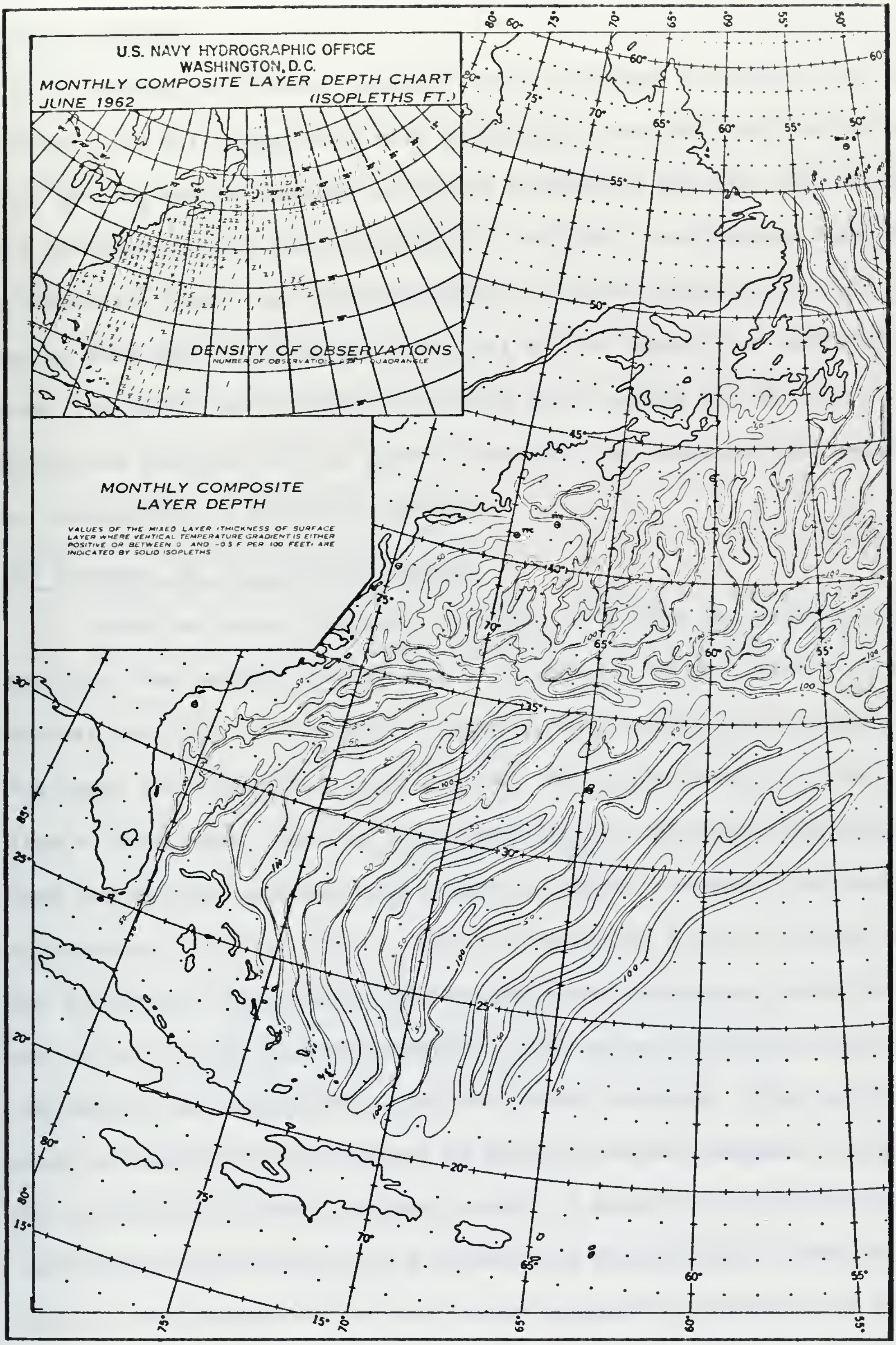

U.S. NAVY HYDROGRAPHIC OFFICE
WASHINGTON, D.C.
MONTHLY COMPOSITE LAYER DEPTH CHART
JUNE 1962
(ISOPLETHS FT.)
DENSITY OF OBSERVATIONS
MONTHLY COMPOSITE
LAYER DEPTH
VALUES OF THE MIXED LAYER (THICKNESS OF SURFACE LAYER WHERE VERTICAL TEMPERATURE GRADIENT IS EITHER POSITIVE OR BETWEEN 0 AND -0.5 F PER 100 FEET) ARE INDICATED BY SOLID ISOPLETHS

FIGURE 11

Optimal Spacing

Although the screen commander has an optimal screening rule for guidance, he will experience some uncertainty over his decision as to what spacing to use because he will be estimating the line efficiency of his possible screens when he makes the decision. For reasons that will be discussed later, the estimated line efficiency function for his screens may be some what in error and hence not be the actual line efficiency function. We define as the optimal spacing that spacing for the actual line efficiency function and the given isoprobability function which minimizes the probability of submarine hitting.

The Accuracy of the Ordered Spacing

When the screen commander attempts to follow the optimal screening rule, the ordered spacing may be considered as an estimate of the optimal spacing. If the screen commander had continuous knowledge of the sonar conditions and could estimate the actual line efficiency function without error, then he would have perfect information in the sense that the optimal spacing would always be known. However, information of this sort does not exist. Thus, as the ships proceed through the sea, the accuracy of the ordered spacing becomes a subjective determination on the part of the screen commander. This accuracy of the ordered spacing implies an uncertainty about the optimal spacing. This uncertainty must be assessed in some manner so that the screen commander can make the decision as to what spacing to use. A convenient form to model this accuracy or lack thereof, is a probability distribution. Thus, we shall describe the uncertainty of the screen commander's estimates of the optimal spacing by a probability density function whose random variable is the optimal spacing.

Cost of BT Information

Because BT information is so important for the screen commander's decision as to what spacing to order, he must be perceptive in recognizing when new information is required. Further, he must appreciate the value of more or current data because the BT information is not free. That is, this information costs something because the screen commander must detach a screening unit from his screen in order to make a BT drop, or in other words, obtain a sample of information. This situation arises because the detached unit usually must proceed more slowly than the screening units while it makes the BT drop. The cost of information then is a reduction in line efficiency which immediately implies some increase in the probability of submarine hitting.

Doctrine

When to obtain the BT information is of critical importance. Most official discussions of when to seek BT information goes as follows:

> The frequency of BT lowering is once each watch (every four hours). However, if charts show ships operating in an area of extreme variable conditions the BT reading should be taken as often as possible.

Obviously no criterion is presented so as to quantitatively or objectively measure when the BT drop should be made. It has been left to the judgment of the screen commander with these almost trivial guidelines.

Cost of Error

The same type of cost that results from a decrease in line efficiency when a ship is detached to make a BT drop will occur when the optimal spacing is not exactly estimated by the screen commander. That is, the probability of submarine hitting will increase if the spacing ordered by the screen commander is in error, and this increase is the cost of error.

Essence of the Decision Problem

It would be convenient to have a function to describe what costs or losses result from errors of various magnitudes in the ordered spacing. How such a loss function is obtained will eventually be pointed out. Further, we will see how this function may be combined with the screen commander's probability distribution function of the optimal spacing to yield an intuitive and consistent decision. Also, we will find out when to obtain a sample of information and how to estimate the expected gain that may result from the additional information. This is the essence of the screen commander's command and control decision problem.

Bayesian Theory

To investigate these interesting areas of the screen placement problem we will use a technique called Bayesian Decision Theory. The underlying structure of this theory is based on Bayes theorem found in probability theory. From the following equivalent statements of the

joint probability of two events H and D,

$$P(HD) = P(H|D)\,P(D) = P(D|H)\,P(H)$$

we can immediately conclude that

$$P(H|D) = \frac{P(D|H)\,P(H)}{P(D)}$$

for $P(D)$ and $P(H) \neq 0$. Thus, if H is some hypothesis such as the screen commander's estimate of the optimal spacing, and D is some report or data such as BT information which contains information about H, the probability of H holding, given that report D is received, is as shown from Bayes Theorem above.

In light of the game theoretic approach which yielded the optimal screening rule, our Bayesian analysis will employ the subjective probabilities of the screen commander to express by these distributions about the optimal spacing, his explicit views of how he feels the submarine commander is evaluating the dynamic decision situation. The screen commander's orderly opinions will be used, not only to evaluate the current estimates of the probability of submarine hitting, or the payoff in the decision matrix, but also to update estimates in the light of new information which will be obtained by conducting a BT drop. This, briefly, is an overview of the Bayesian analysis of the screen placement problem. Hopefully, this Bayesian technique can be quite systematically applied to the naval ASW screen placement problem and provide eventually a means to evaluate proposed procedures in this area.

CHAPTER V

LINE EFFICIENCY

Preliminary Remarks

As indicated in the last chapter, we will now show how information about the sonar conditions of the water and the characteristics of the sonar equipment are combined to yield the line efficiency of a screen. This chapter leans heavily on References (17) and (22) which in turn represent the theory behind the screens which the Navy currently employs. Thus, though large amounts of data may be required to obtain quantities that will be introduced, the efforts are presently being expended in programs such as FADAP, Fleet Antisubmarine Warfare Data Analysis Program.

Separated Glimpses

As we beam our searchlight of acoustic energy at particular points in the sea and then wait to observe any reflected energy, we are obtaining a series of discrete or separated glimpses with which we hope to detect a target if it is present. Let g_i be the glimpse probablility which describes the conditional probability of detecting a submarine on the particular i^{th} glimpse given the submarine is present and has not been detected previously. The index "i" will order a sequence of glimpses which may be made on a submarine as passes a screening unit. This glimpse probability usually varies from glimpse to glimpse, because of factors such as range, aspect of the

target, and environment conditions. Usually as the range increases, the glimpse probability decreases. Thus, as the submarine comes closer to the screening unit the glimpse probability of detection becomes greater. We shall assume at some maximum range, r_m, that the glimpse probability at this range, or greater, is zero. This relationship between glimpse probability and range can be summarized generally in the following figure (Figure 12) where r_i is the range on the i^{th} glimpse.

If the screening unit has a finite number of glimpses, say n, at the submarine, then the probability of detection in this n glimpses, denoted F(n), is equivalent to one minus the probability of not detecting in the n glimpses, or

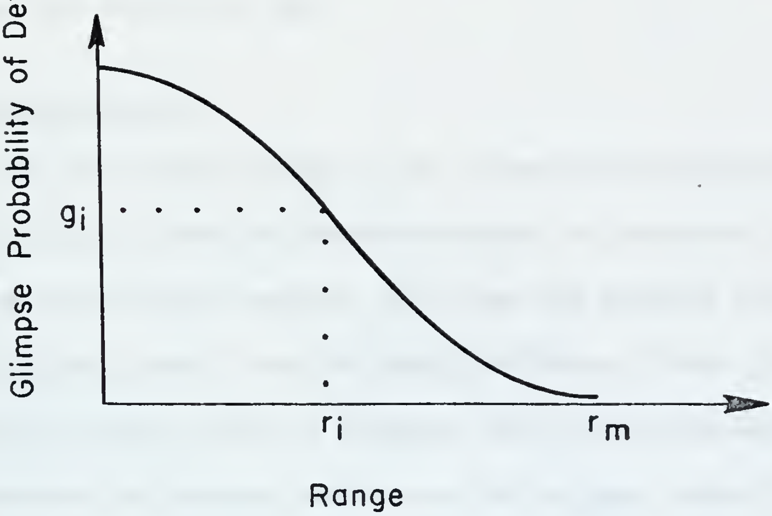
$$F(n) = 1 - \prod_{i=1}^{n} (1 - g_i) \qquad (5\text{-}1)$$

This expression will be useful in later discussion.

<u>Detection Zone and Lateral Range</u>

A circle of radius, r_m, circumscribed about a screening unit will describe a region in which a submarine can possibly be detected if it is present. Thus, detection of a submarine is possible only if the relative motion between the screening unit and the submarine brings the submarine through this region called the zone of possible detection.

As the submarine moves through the zone of possible detection of a screening unit it traces a line of relative motion. The range to the submarine at its closest point of approach to the screening unit is defined as the lateral range. It describes the position of a



Glimpse Probability Versus Range

FIGURE 12

particular line of relative submarine motion with respect to the screening unit. Let us note that the lateral range is a physical parameter. It will be denoted by the symbol x. The following figure (13) illustrates the kinematic situation between the screening unit and the submarine, where Z_1 is the point of entry and Z_2 the point of departure from the detection zone.

Lateral Range Curve

As the lateral range of the submarine increases, not only does its line of relative motion through the detection zone of a screening unit become smaller, but also the glimpse probabilities become smaller, due to the increase in range. Thus, for a fixed glimpse rate, the number of glimpses that the screening unit can take at the submarine becomes smaller as the lateral range of the submarine increases. Hence, for a particular lateral range there exists a cumulative probability of detection, denoted P(x), which can be determined from the expression for the probability of detection in n glimpses. That is,

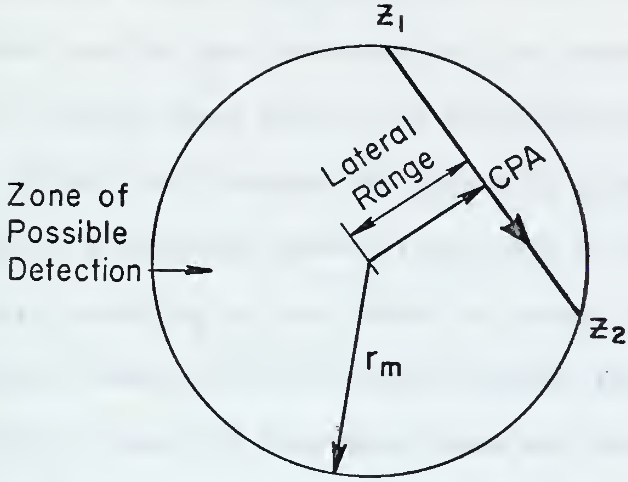
$$P(x) = F(n) \qquad (5\text{-}2)$$

or,

$$P(x) = 1 - \prod_{i=1}^{n} (1 - g_i) \qquad (5\text{-}3)$$

where n and g_i's are known.

A function which gives the probability of detection of a submarine which passes at some lateral range is called the lateral range curve. This curve is typically represented as a symmetric curve about



Kinematics of Detection

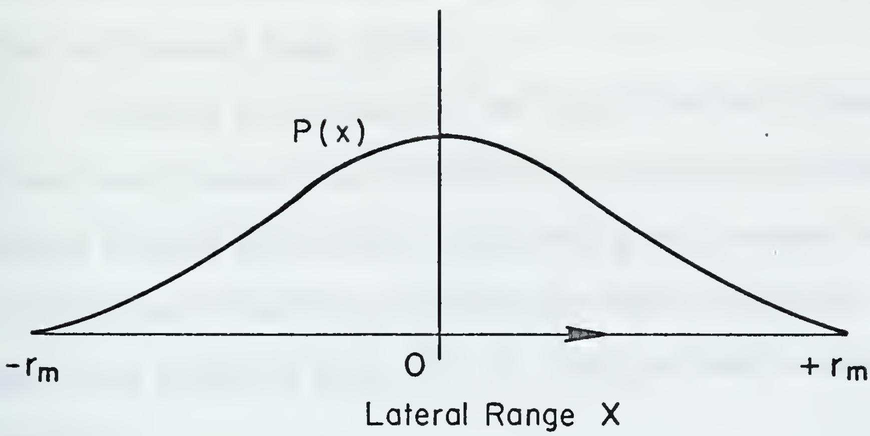
FIGURE 13

the screening unit, from the maximum range on one side to this same range on the other side of the screening unit. The curve is illustrated in Figure 14.

The lateral range curves for screening units are very important in antisubmarine warfare because they are used to summarize the detection capability of the screening unit. However, in order to use these curves, we must be aware that environmental conditions of the water, type of target, the particular sonar, and the type of scanning procedures used by the screening unit to obtain its glimpses will vary the lateral range curve of a screening unit. This latter condition is significant because the number of glimpses a screening unit may take of a submarine transversing with a particular lateral range will vary, depending on how often the acoustic beam is pointed in its direction. However, we can usually assume that the scanning effort is uniformly conducted throughout some arc about the screening unit.

We have already pointed out that the water conditions will affect the path of the sound energy and that sonars operate at different frequencies with various power limitations. Thus, for different water conditions and sonar equipment we will obtain different lateral range curves.

The type of submarine will also vary the lateral range curves because an increase in the size of the target increases the glimpse probability and the speed of a target will also affect the number of glimpses the screening unit may take.



Typical Lateral Range Curve

FIGURE 14

Because submarines have different maximum limits on the pressure they can withstand against their hulls, the maximum depth at which the submarine can hide will vary. Thus, if a submarine can dive below the isothermal layer, as is the usual case, it will be able to hide more effectively than if it had to remain in the layer. Thus, depth of the submarine will also be an influential factor in determining the lateral range curve.

In order to account for the almost infinite number of possible lateral range curves, the influential factors are currently grouped with one lateral range curve representing the average conditions within each group. Thus, for a particular sonar, operating against a small, slow target in high seas, we will use some average lateral range curve.

Throughout this analysis we assume that the lateral range curves for the various groupings are known to the screen commander. Thus, when he estimates the type of submarine that poses the major threat and measures the BT conditions of the water, he then chooses the corresponding lateral range curve for the submarine at best depth to avoid detection. We might note here that tables and graphs are presently available and used to provide similar information to that which we assumed above.

The Analytical Description of Line Efficiency

We shall now use the lateral range curves to develop an analytical expression of the line efficiency of the screen as a function of the screen spacing. Let us take two screening units, A and B, with some spacing, s, apart and consider a submarine penetrating through the screen so as to have a lateral range, x, from screening unit A. The probability of detecting this submarine as it transverses the zones of possible detection, given the spacing between units and the lateral range from A, denoted $P(D|s,x)$, is equal to the probability of detecting the submarine by either unit A or unit B, or both. For active sonar independence is assumed between the events of detection by unit A and detection by unit B. Hence, we have the expression for calculating this probability as follows:

$$P(D|s,x) = P_A(x) + P_B(s-x) - P_A(x)P_B(s-x) \qquad (5\text{-}4)$$

where $P_A(\cdot)$ and $P_B(\cdot)$ denote the probability from the appropriate lateral range curves for screening units A and B, respectively.

Recall that line efficiency is the probability of detecting a submarine penetrating the screen with some given spacing. Therefore, line efficiency can be expressed in terms of the probability of detecting a submarine penetrating the screen with a given lateral range and spacing and the probability density function that describes the chance for a particular lateral range to occur, denoted f(x), the source of which must be screen commander. When f(x) is continuous, the line

efficiency can be expressed as follows:

$$LE = P(D|S) \tag{5-5}$$

or,

$$LE = \int_{x} P(D|S,X)\ f(x)\ dx. \tag{5-6}$$

For the case when the screening units have the same sonar equipment and the screen commander feels that penetration is nearly random so that all possible lateral ranges of the submarine are equally likely, the line efficiency can be computed using a uniform distribution for f(x) to yield the following expression:

$$LE = \frac{1}{S} \int_{x=0}^{S} \left[2\ P_A(x) - P_A(x) \cdot P_A(x-s)\right] dx \tag{5-7}$$

If the line efficiency is to be the same between each pair of adjacent screening units so that the screen commander may evoke the optional screening rule, then either of two cases must exist. First, if all screening units have the same type of sonar equipment which implies the same appropriate lateral range curve for each, then the same spacing between each pair of adjacent screening units will result in a line efficiency that will consistently measure the probability of detection of a submarine penetrating through any point on the screen. For the second case, when there are different detection devices present, the spacing between each pair must vary to yield the same line efficiency between each pair. For simplicity and to match what is currently being practiced, the assumption will be made that

all screening units will have the same lateral range curves. Thus, as we have already seen, spacing between units will be used as the controllable decision variable of the screen commander. Note that even if the detection devices were not the same that the decision variable may be either a function of different combinations of spacings between units or the distance from the main body. The possible complexity of computation is ignored by assuming the same lateral range curve for each screening unit.

We have now seen how the information about sonar conditions, estimate of the threat, and other variables are combined through lateral range curves to yield an estimated line efficiency for a screen with a particular spacing. We must now turn our attention to the task of developing or structuring a measure that will describe the cost incurred by the screen commander when the spacing he chooses to order for the screen is not the optimal spacing.

CHAPTER VI

LOSS/GAIN STRUCTURE

Loss Function

The development of the screen commander's estimate of the line efficiency of his screen as a function of spacing has been presented in the last chapter. Because of the possibility of changing water conditions, grouping of lateral range curves, and the other complex characteristics associated with the screen placement problem, we can see that the estimated line efficiency for a particular spacing need not be an exact or accurate representation of the actual line efficiency corresponding to the same spacing. This actual line efficiency represents the true state of nature and any deviation of the screen commander's estimate from the actual line efficiency will create a problem when he follows the guidelines of the optimal screening rule, and lead to some increase above the minimum probability of submarine hitting.

An interesting question that suggests itself now, is whether or not the submarine commander knows the actual line efficiency function. We will circumvent this question by assuming that the screen commander acts as if the submarine commander does possess rather accurate knowledge about the line efficiency of the screen. As will be shown later, if the submarine commander acts without any such knowledge, it will only lessen his chance of hitting the main body. However, the reasonableness of the assumption stems from the fact that the submarine is located in the depths of the water and with the continuous BT information that it possesses is able to go to the best depth to avoid detection. Combined

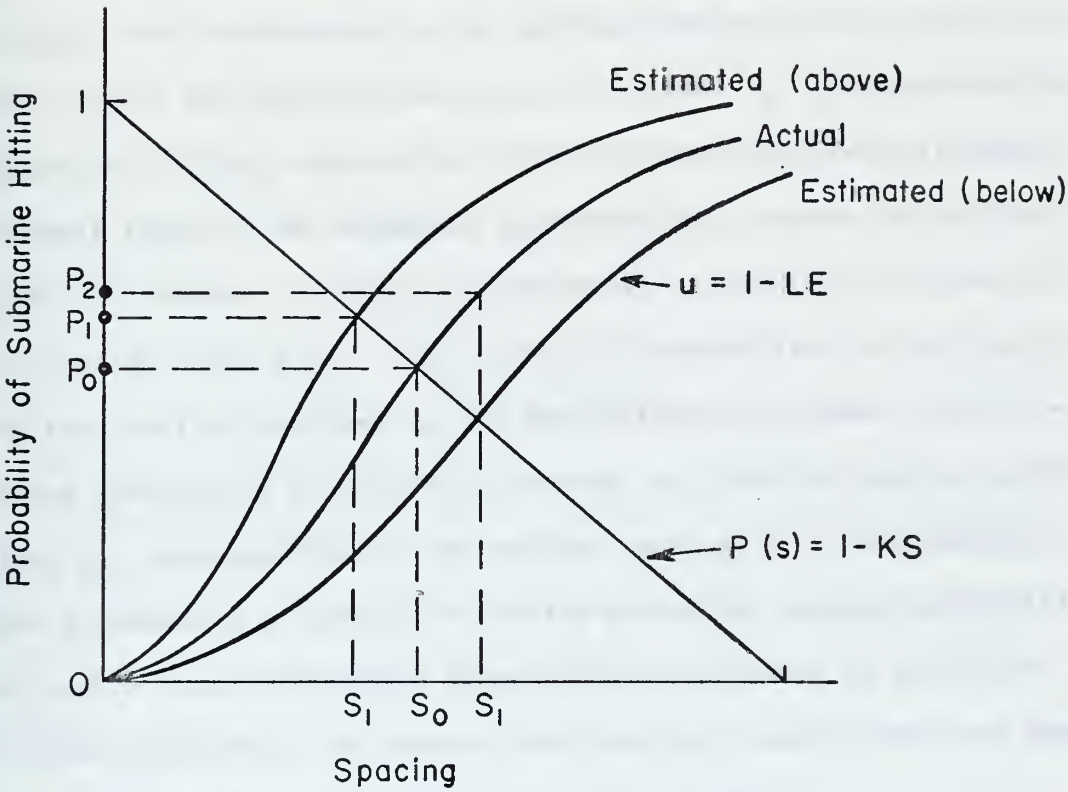
with the idea that the submarine commander knows the characteristics of the sonar of the screening units - which is similar to the assumption the screen commander knows the characteristics of the weapons of the submarine - this assumption becomes tenable.

Now let us revise Figure 9, which illustrated the optimal screening rule by using two curves to represent line efficiency. One will correspond to the actual line efficiency function and the other to the estimated line efficiency function. Of course, the estimated line efficiency may lie above, on, or below the actual line efficiency function. These cases are illustrated in Figure 15.

If the estimated line efficiency function were in fact equal to the actual line efficiency function, and the optimal screening rule was followed, then the spacing chosen by the screen commander would indeed be the optimal spacing which we defined previously. Hence no possible decrease in the probability of submarine hitting could be made by the screen commander's choice of any other spacing. The line efficiency that corresponds to this optimal spacing will be denoted by LE_o and the corresponding probability of submarine hitting by p_o. This notation leads to the following expression:

$$p_o = 1\text{-}LE_o \qquad (6\text{-}1)$$

From an examination of Figure 15 we can see for the case when the function representing one minus the line efficiency function, say, undetected or successful penetration function, lies above the



Losses from Non-optimal Spacing

FIGURE 15

actual penetrating function then the spacing, s_1, used by the screen commander will be smaller than the optimal spacing. Recall that the submarine commander will choose the maximum of the probability of submarine hitting represented by the value of the linear contour function, or the actual undetected penetrating function, denoted by $1 - LE_{ACT}$, that corresponds to the spacing ordered by the screen commander. Thus, for the case when s_1 is less than s_o, the probability of submarine hitting represented by the contour function will always be chosen; that is, the submarine commander will choose to fire from outside the screen. Let this corresponding probability of submarine hitting be denoted by p_1. The increase in probability of hitting by using this smaller spacing, s_1, is the difference between its corresponding probability of submarine hitting, p_1, and the similar probability, p_o, corresponding to the optimal spacing s_o. Any increase in the probability of submarine hitting above the minimum probability that results when the optimal screen rule is followed is a loss to the screen commander. We denoted this loss by L and in the case under consideration, the loss is given by the following:

$$L = p_1 - p_o \tag{6-2}$$

When using similar notation as above where

$$p_1 = 1 - LE_1 \tag{6-3}$$

we obtain

$$L = (1 - LE_1) - (1 - LE_o) \tag{6-4}$$

or

$$L = LE_o - LE_1 \tag{6-5}$$

For the case when the undetected penetrating function lies below the actual function, the screen commander's ordered spacing, say s_1, will be greater than the optimal spacing. In this case, the submarine commander will choose to penetrate the screen because the probability of submarine hitting, represented by the actual undetected penetration function, is greater. Letting p_2 represent this probability and designating $1\text{-}LE_{ACT}$ by $1\text{-}LE_2$, we see that the following expression holds,

$$p_2 = 1 - LE_2 \; . \tag{6-6}$$

The loss in this case where s_2 is greater than s_o is given by the following:

$$L = p_2 - p_o \tag{6-7}$$

or, we can write

$$L = (1 - LE_2) - (1 - LE_o) \tag{6-8}$$

or

$$L = LE_o - LE_2 \; . \tag{6-9}$$

We can summarize the above cases by writing the loss function that denotes the increase in probability of submarine hitting for some deviation of ordered spacing from the optimal spacing as follows:

$$L = \begin{cases} p_1 - p_o & s \leq s_o \\ p_2 - p_o & s \geq s_o \end{cases} \tag{6-10}$$

Or, we can write

$$L = \begin{cases} LE_o - LE_1 & s \leq s_o \\ LE_o - LE_2 & s \geq s_o \end{cases} \qquad (6\text{-}11)$$

We have already agreed that the screen commander's objective is to minimize the probability of submarine hitting, thus if we minimize the screen commander's loss, the objective will be optimized. However, if an economic analysis is performed with loss function, L, to determine what spacing the screen commander should order, then results are obtained that require unmeasurable quantities, such as the actual line efficiency function. Thus, it is impractical to use. Nevertheless, we may conceptually treat the actual line efficiency function as being known, in order to investigate a deviation between the actual and estimated line efficiency functions. Thus, we again turn to Figure 15 and formulate a new function, called the gain function. Fortunately we will see that the use of the gain function eventually allows impractical difficulties, associated with the loss function, to be overcome.

Gain Function

Using Figure 15, let us observe that the closer we can make the estimated line efficiency function to the actual line efficiency function, the smaller the loss will be for some ordered spacing that is not the optimal spacing.

In order to simplify this observation, we denote the value of the actual successful penetration function at the optimal spacing s_o by $(1-LE_{ACT} 1 s_o)$ and the value of the estimated penetrating function

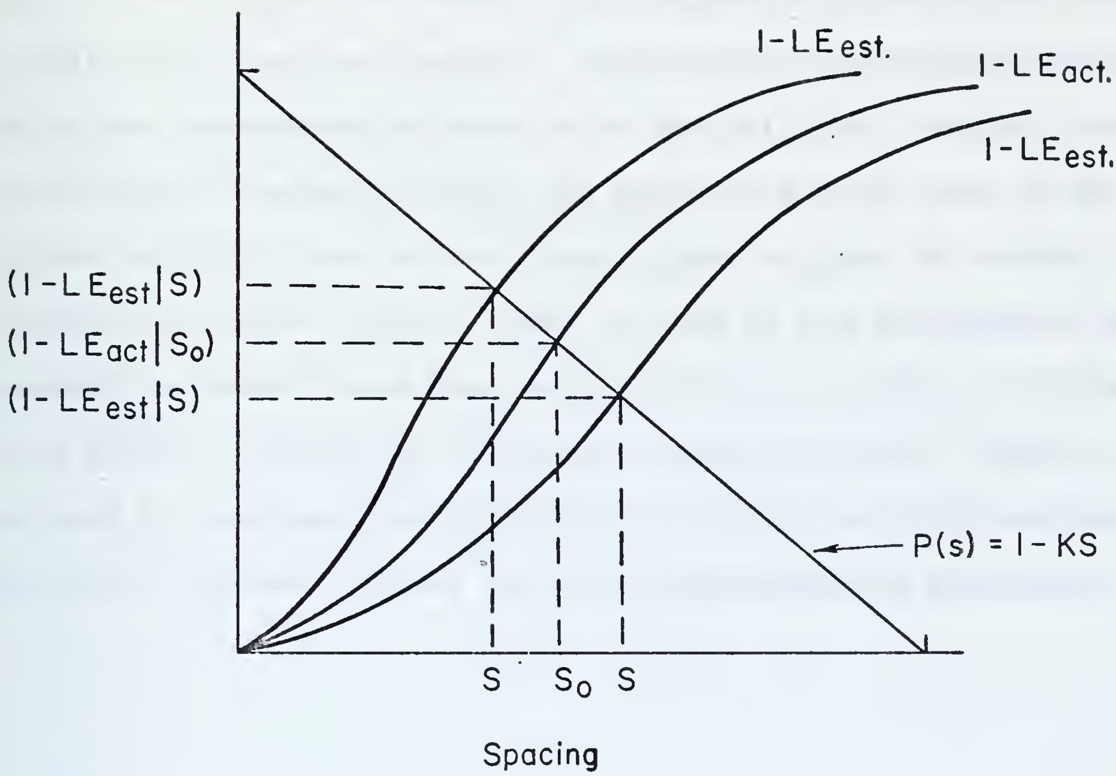
at the screen commander's choice of spacing s by $(1\text{-}LE_{EST}1s)$.

We will now examine two cases. The first instance is when the screen commander's choice of spacing, as he follows the optimal screening rule, is greater than the optimal spacing or, in other words, the estimated penetrating function is below the actual penetrating function. The second case is when the screen commander's choice of spacing is less than the optimal spacing or when the actual penetrating function is below the estimated penetrating function. Revising Figure 15 to Figure 16 to conform with the notation just introduced, let us notice that for the first case, if the screen commander's choice of spacing is such that the difference of $(1 - LE_{ACT}\ 1s_o)$ less $(1 - LE_{EST}\ 1s)$ is as small as possible, then the less will be the loss as determined by the loss function that was just developed. For the second case, we can observe that if the screen commander's choice of spacing is such that the difference of $(1 - LE_{EST}\ 1s)$ less $(1 - LE_{ACT}1s_o)$ is as small as possible, then again the minimum loss will result from the screen commander's choice. Of course, if $1\text{-}LE_{EST}$ is equal to $1\text{-}LE_{ACT}$, the choice will indeed be the optimal spacing.

A function, denoted g(so), which suggests itself from these observations is the following:

$$g(s_o) = \begin{cases} (1 - LE_{ACT}1s_o) - (1 - LE_{EST}1s) & s_o \leq s \\ (1 - LE_{EST}1s) - (1 - LE_{ACT}1s_o) & s_o \geq s \end{cases} \qquad (6\text{-}12)$$

Therefore, if the screen commander chooses his ordered spacing so as to minimize the function g (s_o), which we shall call our gain function, he will realize the minimum probability of submarine hitting or the



Probability of Submarine Hitting Versus Spacing

FIGURE 16

maximum payoff relative to the screen commander's decision matrix Dp. Let us notice that we can write expression 6-12 for the gain function as follows:

$$g(s_o) = \left|(1 - LE_{EST}1s) - (1 - LE_{ACT}1s_o)\right| \qquad (6\text{-}13)$$

and in turn, choose s to minimize this expression to achieve the same results as we have just obtained. Interestingly this expression implies some minimum deviation may be of interest to us. However, both expressions of the gain function are conditioned on the value of the optimal spacing s_o and we have already agreed to treat the optimal spacing as a random variable. Thus, we need to turn our attention to probability theory, Bayes Theorem in particular, in order to find the best choice of spacing for the screen commander to order. That is, we need to investigate the probabilistic phase of our study now that the screen placement problem has been deterministically structured.

CHAPTER VII

PROBABILISTIC INFORMATION PROCESSING SYSTEM

Fundamental Concept

We have already indicated there are several sources of information with which the screen commander may deal. Further, we have recognized that uncertainty is a characteristic that is unyieldingly associated with the screen placement problem. We shall now attempt to manipulate the information and uncertainty by modeling the command and control decision process of the screen commander as a probabilistic information process, denoted PIP. Edwards (3) first proposed PIP in 1962, as an aid to making diagnostic or command and control-type decisions. The fundamental concept of a PIP system is to use expert human judgment in order to process fallible information probabilistically so that some quantitative indication of its reliability can be obtained. An acceptance of the military to use PIP systems to model command and control processes would be revolutionary, because as Edwards et al (6) points out, the systems now used, process fallible information deterministically in that information that is filtered via some aspiration level of relevance, is handled with no quantitative indication of its fallibility. A question which suggests itself immediately is: Can the military handle information more effectively than it is presently doing by deterministic methods so that the reliability of the information can be indicated?

Statistics and Information Processing

Statistics has long been used to summarize and manipulate information or data. However, because the school of thought which required a long-run frequency interpretation of probability prevailed among the statisticians, an effective use of statistics was not accomplished in the area of human decision making in military command and control systems. This result was due to the fact that most command and control systems deal with rapidly changing environmental processes describable by a variable array of attributes, but not readily describable in frequentistic terms. However, the barrier presented by the non-frequentistic environments has been penetrated by the recent emphasis on the use of Bayesian statistics in decision theory. Rather elementary presentations of the advantages of Bayesian statistics that is oriented to stress its relevance to military information processing is presented in two articles by Edwards et al, (5) and (7), which report results of complex experiments that investigated military command and control systems. However, these works only investigated how human judgment or opinions were revised in the light of new information. Our analysis of the naval screen placement problem differs from Edward's work in the sense that the probabilistic information processing system is incorporated with our loss/gain structure so that the economic or beneficial advantage of new information, measured by a change in the probability of submarine hitting, can be realized by the screen commander. This incorporation of a PIP system and payoff function makes our decision-making model complete in the sense

of Howard's claim that "the essence of decision making is understanding the economic impact of uncertainty." (13-54). Thus, placing a value on the reduction of uncertainty is the first step in our Bayesian approach to the screen placement problem. We have developed the loss/gain structure for this reason, because only when the screen commander knows what it is worth to reduce uncertainty does he have a basis for detaching a screening unit to obtain information by conducting a BT drop--whose purpose is to help reduce uncertainty. Further, in our command and control system, the screen commander must decide which spacing is most plausible to order as his estimate of the optimal spacing. The use of a PIP system and an economic measure to indicate the payoff related to various possible spacings combines payoff with plausibility, so that the screen commander can evoke the criterion of minimizing the probability of submarine hitting in order to make a best choice or decision with the assistance of Bayesian logic.

Bayesian Statistics

One of the fundamental concepts of Bayesian statistics is that probabilities are defined as consistent orderly opinions of a decision maker. The importance of human judgment is emphasized by this definition of probability. Further, we can see that our previous representation of the description of the uncertainty of the screen commander's estimate of the optimal spacing by a probability density function qualifies this probability as a Bayesian probability. Any probability that is mentioned from this point on will be a personal or Bayesian probability of the decision maker.

Basically, Bayesian statistics is concerned with processing new information so that previously-held opinions may be revised in the light of this additional information. Because all too often the information used for decision making is fallible, imperfect, or uncertain in the sense that it may be highly in error, Bayesian statistics uses Bayes Theorem to process this fallible information in order to aid human judgment by showing how new observations or data should change or reinforce previously-held opinions. From the logic of the theory of mathematical probability, Bayes theorem is shown to be the formally optimal or appropriate rule for modifying the probability that a hypothesis is true when new evidence has come forth.

Information Processor

The main objective of a PIP system is to get maximum benefit from available data as possible when making decisions. For this reason, Bayes theorem is used as an optimal model for processing information in military command and control decision situations. In particular, to our naval screen placement problem we shall use the screen commander as an information processor in that he will supply certain inputs, using his personal logic or intuition, to an information processing procedure or mechanism which will use Bayesian logic to process the information so that it may be examined in light of the screen commander's decision rules or guides.

Subjective Probability Approach

Because we have defined probability as a personal orderly opinion on the part of the decision maker, we use it to measure the confidence that he has in the truth of a particular hypothesis. The only restrictions on our subjective quantification is that it obey certain conditions so that the numerical weights of conviction may be manipulated according to the mathematical theory of probability. These restrictions are the basic axioms of the mathematical theory of probability and are as follows:

1. The probability of an event is a number that must be greater than, or equal to zero, or less than or equal to one.
2. The probability of a collection of exhaustive events is equal to one.
3. The probability of either of two mutually exclusive events must equal the sum of the probabilities of each event.

The subjective probability approach to decision making opens new frontiers of problems which previously could not be formulated and makes them vulnerable to being solved. It must be emphasized that the utility of subjective probability is that it requires the use of human judgment and experience of the decision maker in order for him to order his opinions or assign weights that reflect his ventricular feelings about some set of propositions.

Probability Assessments

In military situations which deal with non-frequentistic environments, it is the interaction of judgmental factors in the decision maker's mind that is assessed by his personal probability estimates. A concern that suggests itself now is the existence of some criterion to judge probability assessments. All consistent, orderly assessments are allowed as long as the decision maker feels they represent his judgment. Winkler (27-766) offers a criterion that indicates a probability assessment should be considered good if it corresponds both to the judgment of the assessor and to reality or the description of the actual decision situation. In military applications this criterion will most likely be met when the expertise is the assessor.

Recall that we have stated Bayes theorem as follows:

$$P(H|D) = \frac{P(D|H)\ P(H)}{P(D)} \tag{7-1}$$

where P(D) and P(H) are not equal to zero. The probability $P(D|H)$ is the likelihood that the decision maker will receive a particular set of of data if the hypothesis under consideration is true. Assessing weights that can be used as $P(D|H)$ is the key to using Bayes theorem in a PIP system. It is the most important step in applying Bayes logic to a military command and control decision problem. Thus, it is to the commander with his valuable experiences that we turn to obtain such vital assessments.

In the screen placement problem we shall suppose that the subjective opinions elicited from the screen commander and based hopefully on pertinent naval experience will be such that they readily describe the information gleaned from actual observations of the decision situation. Or, in other words, we will treat the probability estimates as pertinent quantitative measures in our mathematical models because they are assumed to meet our acceptable criterion of goodness.

We must again realize that we are dealing with an intuitive process or cognitive one rather than a perceptual one. Thus, the screen commander must form his probability estimates over many attributes rather than on a single perceptual continuum such as distance to a target, or the height of a signal on a cathode ray tube. We are using human judgment to process any lack of independence among the various attributes and to integrate the output of different data sources into a single orderly estimate. Hence, more personal or subjective judgment is involved. Hopefully, the use of the theory of kinetic search to structure an analytical presentation of the screen placement problem will appropriately influence the screen commander's personal logic and help to identify relevant factors, so that his subjective intuition will change with experience in such a manner that will be beneficial to producing a more efficient solution to the screen placement problem. Thus, rather than to handle one value of Effective Sonar Range - a particular function of the lateral range curve of the detection equipment - by a deterministic or fixed method in order to arrive at a screen placement solution, as is done

today, the screen commander may consider his total non-frequentistic environment. This latter consideration must take place because the use of a PIP system requires the screen commander to express his uncertainty or lack of confidence in the ordered spacing of his screening units. Thus, the screen commander can process fallible information and identify its relevance to the screen placement problem.

It should be stated here that the motivation of probabilistically assessing expert judgment that reflects reality is in its being an extension of the "main assumption of the philosophy that accompanied the development of the scientific outlook, that the real may be identified with the quantitative." (24-135) So be it.

Potential of PIP

The question that may still arise is why a PIP System? Or, in other words, why not assess the probability $P(H|D)$, that is obtained from our use of Bayes theorem, directly from the screen commander? Extensive studies, some by Edwards et al (5, 7) have demonstrated that a typical decision maker wants to be more certain than is necessary and thus attempts to accumulate too much information. However, intellectual and not motivational deficiencies are suggested as possible explanations for this result. Thus, a command and control system which takes advantage of the Bayesian logic of the PIP system will be more effective than other information processing systems.

Studies have been conducted by Kaplan and Newman (13) which offer a criterion to measure the effectiveness of a PIP system.

Their result is that a Bayesian information processing of human judgement is more effective in the sense that less information is required to arrive at the correct decision. These same studies also suggest that a PIP system would be most applicable to those occasions for which decisions must be made quickly and on the basis of small quantities of information. Their claim is that if there is opportunity to obtain large amounts of relevant data over an ample sequence of time, the advantage of processing information via Bayes theorem lessens. Let us note that due to the action or time pressure, and the small amount of data, a single BT drop, involved in the screen placement problem, it becomes clear that the screen commander's decision as to what spacing to choose is a candidate for being modeled as a probabilistic information processing system - the probability distributions of which are our next concern.

CHAPTER VIII

PROBABILITY DISTRIBUTIONS

Preliminary Remarks

We have seen, because the screen commander must somehow integrate BT information relating to sonar conditions, present performance of detection devices, estimates of the potential of the submarine weapons system, sea state, wind, and other significant factors related to the complex ASW detection problem, that uncertainty is persistently interwoven in the screen commander's decision problem. We have elected to express this uncertainty by using the language of probability theory and to revise any change in the screen commander's uncertainty in accordance with the logic of Bayes theorem. It is our purpose now to specify the probability distributions required for our PIP system and to examine how to use them.

Prior Distribution

We shall suppose that the uncertainty related to the screen placement problem of any screen commander can be modeled probabilistically. We have already decided to treat the optimal spacing, s_o, as a random variable in order to describe the screen commander's uncertainty and as such, elicit an orderly opinion from the screen commander about possible values of the optimal spacing. The very special expression of information of interest now is really the experience or total knowledge of the screen commander at the initial stage of a decision

before any information about the current water conditions is made available. This subjective probability distribution function that describes the screen commander's belief about the present values of the optimal spacing for his screening units is the function known in Bayesian Decision Theory as the prior distribution of s_o, it being made explicit prior to receiving current sample information such as the report from a BT drop. The prior distribution will be denoted by $PR(s_o)$.

Even though the language of probability may be unfamiliar to a screen commander when he is initially introduced to the PIP system, he should be made aware that his elicited prior distribution may be considered as conceptually equivalent to the current practice of ordering a specific screen spacing before his screening units get under way from a port and are able to obtain BT information. That is, the screen commander continues to use his prior experience in decision making though we now attempt to make his decision process more explicit by eliciting his prior distribution in a quantified form. We must now heuristically develop an adequate mathematical function that can represent the prior distribution.

We shall now separate the various factors that influence our complex ASW problem into a dichotomy of uncertainty and show that for each case a probability density function for a given value of screen spacing can be used to describe it. On one hand, we consider the screen commander's lack of knowledge about the potential of the

submarine's weapons to be the main source of uncertainty that influences the screen commander's feeling about the locations of the contours of equal probability of submarine hitting the main body. On the other hand, we treat estimates of the sonar conditions, supposed performance of screening units, environment, and so forth, as the resolving factors over the screen commander's uncertainty about the line efficiency of his screen. Recall that we have already shown that a special combination of the linear isoprobability function and the penetrating function determines the optimal screening rule. Thus, the combining of the above two cases to produce a description of the uncertainty of the screen commander about the optimal spacing naturally suggests itself. Thus, for various spacings, the screen commander should be asked for some prior estimates of both the probability of submarine hitting, given he fires just outside the screen, denoted $PR(p|s)$, and the probability of successful penetration, if the submarine were to attempt to penetrate, denoted $Pr(q|s)$. Along with these estimates we will, of course, want some range within which the screen commander is assured the values that he is estimating will fall. That is, we want some indication of the reliability he holds in his subjective feelings.

For the case in our dichotomy of uncertainty that is related to the submarine weapons system, it seems reasonable that the screen commander would feel more assured that his estimates of $PR(p|s)$ would be very close to one and zero for a very small spacing and a very

large spacing respectively. We can represent the assurance of the screen commander's feelings by using the variance of the probability density functions being elicited to indicate the relative uncertainty over the set of possible outcomes of $PR(p|s)$. Because we are interested in the screen placement problem which has been presented as occurring after the estimate of the threat, a reasonable manner by which the commander may assign his prior estimate for the case being discussed is to let the mean of his $PR(p|s)$ equal $p(s)$ from the estimate of the threat. Thus, if we let $PR(p|s)$ be distributed with a mean, denoted m_H and variance, say V_H, and use the given estimate of the threat, we have

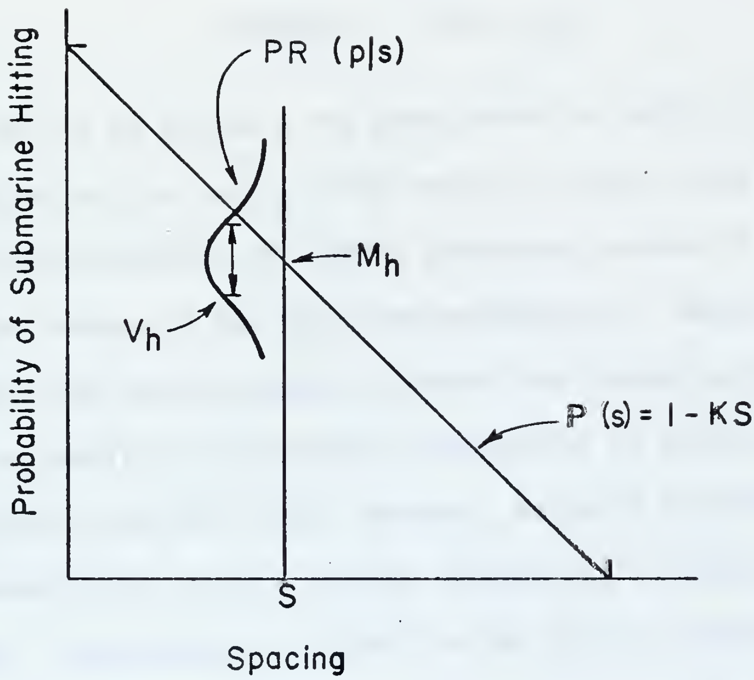
$$PR(p|s) = F(m_H, V_H) \qquad (8\text{-}1)$$

where

$$m_H = p(s) = 1 - ks \qquad (8\text{-}2)$$

Thus, we take the estimate of the threat as given but now allow any uncertainty which the screen commander may feel towards it enter by means of the variance, V_H; for his estimates corresponding to a very large or small spacing he would represent his assurance with a small value of V_H. We can summarize our discussion of this case of the dichotomy of uncertainty with Figure 17.

Now, we examine the remaining case of our dichotomy, that one associated with detection or line efficiency. Because it will be more convenient to look at the undetected penetration function, as we have already hinted, we will recall that the penetrating function is



Subjective Probability of Submarine Hitting for a given Spacing.

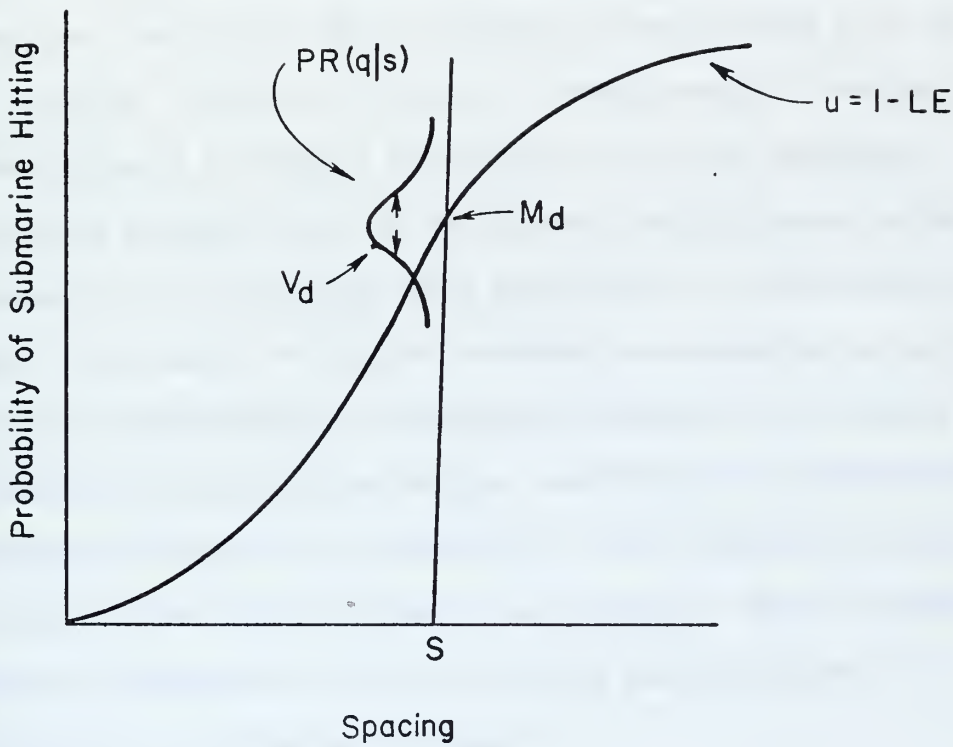
FIGURE 17

one minus the line efficiency function. Thus, we now desire the screen commander to summarize his personal feelings about the probability of successful penetration for some given screen spacing. This we have already denoted $PR(q|s)$ and will denote its mean by m_d and variances V_d. Thus, we have the following expression:

$$PR(q|s) = F(m_d, V_d) \tag{8-3}$$

Note now that we do not have any ready guide by which to assign a value to m_d as we did for m_H. The reason for this stems from the fact that the essence of the screen placement problem is the complex and changing nature of the detection probability. However, we can observe that the screen commander should feel rather assured, not only that the probability of successful penetration is almost zero, if his screening units are very close together, but also that this probability is close to one, if the spacing between his screening units is very large. The variance assigned for the prior estimates of these spacings would be small. We can thus reflect the screen commander's personal feelings and assurance due to the complex detection factors by the prior probability density function for a given spacing as shown in Figure 18.

We now want the screen commander to implicitly integrate together in his mind, the uncertainties of our dichotomy in light of the optimal screening rule, so that his combination of the uncertainties can be summarized by a single expression of uncertainty about



Subjective Probability of Successful Penetration for a given Spacing.

FIGURE 18

the possible choices of the optimal spacing. This expression is our prior distribution of the optimal spacing.

Proceeding with our heuristic development, we must realize that as the estimates of the probabilities associated with our dichotomy begin to converge on the same probability of submarine hitting that the given spacing for these estimates is more likely to be the optimal spacing. Further, if there is a small range of spacing over which the estimates of $PR(p|s)$ and $PR(q|s)$ are nearly equivalent, then the prior variance, V_{pr}, of the optimal spacing should be small and conversely. The following model should help to conceptually clarify what we are asking the screen commander to accomplish as we elicit his prior distribution of the optimal spacing. Let us assume that we are given the distributions, $PR(p|s)$ and $PR(q|s)$ for various spacing and that we would like to find $PR(s_o)$. For simplicity, we will assume that all the random variables are discrete. Thus, following the optimal screening rule, we can write the probability that a given spacing s is optimal as follows:

$$\text{Prob (s is optimal)} = \text{Prob}\ (\ (p|s) = (q|s)\ ) \tag{8-4}$$

$$= \sum_{(p|s)} PR(q = p|s)\ PR(p|s) \tag{8-5}$$

$$= \text{Prob}\ (s = s_o) \tag{8-6}$$

$$= PR(s_o) \tag{8-7}$$

This model reflects the intuitive process that should take place in the mind of the screen commander. An immediate result of this model

is that because of the variances V_H and V_D are small, for both a large and small value of given spacing that the corresponding prior probability of either of these spacings being optimal, is small. Thus, any elicited prior distribution should be such that it shows a small probability for the values of spacing corresponding to the tails of the probability density function. An assumption that we shall now make which agrees with this observation is that the prior probability density function can be modeled explicitly as a normal probability density function. We can summarize this idea symbolically as follows:

$$PR(s_o) = N(m_{pr}, V_{pr}) \tag{8-8}$$

where $N(m_{pr}, V_{pr})$ denotes a normal probability distribution with prior mean m_{pr} and prior variance, V_{pr}. We shall ignore the fact that the range of a normal probability density function is from minus infinity to plus infinity because we have observed that any reasonable variance that reflects the screen commander's subjective assurance will most likely assign to negative values or extremely large values of spacing, a measure of probability that is negligible.

We can check for consistency in the screen commander's intuitive process by discussing the estimates of $PR(p|s)$ and $PR(q|s)$ for some spacings and then directly attempt to elicit the spacing which the screen commander feels is the most likely the value of the optimal spacing, our m_{pr}, and then asking for a variance to once again indicate the reliability of his estimate. Any disagreement that may arise when these two procedures are followed should be pointed out

and freely discussed with the screen commander to identify the cause of the discrepancy.

To summarize this discussion of the prior distribution of the optimal spacing, we assume that this distribution can be elicited from the screen commander and modeled as a normal distribution with prior mean m_{pr} and prior variance V_{pr}.

Information

We have already mentioned that the relative uncertainty among the possible values of a random variable such as optimal spacing is indicated by the variance of the probability distribution. That is, for two probabilistic descriptions of a random process, the one with the smaller variance implies to a decision maker, a small relative uncertainty about the outcome of the random process. Because a decision maker usually obtains more data in order to lessen his uncertainty, we will measure the amount of information in any data by the amount of change in uncertainty that results from the data. Further, we quantify this measure of information in an expression of a normally distributed random process, and this is the case of interest in our problem, by using the reciprocal of the variance to indicate the amount of information associated with the expression. This is a concept first proposed by R.A. Fisher and is appealing because as variance decreases, the amount of information increases and of course the relative uncertainty decreases. This measure of the amount of information will permit us to illustrate how subsequent Bayesian results can be interpreted to agree with one's "common sense."

Likelihood Distribution

The next probability distribution that we elicit from the screen commander for use in our PIP system is called the likelihood distribution. It is the probability $P(D|H)$ whose importance was stressed in the discussion of the PIP system. The likelihood distribution reflects the use of the screen commander's judgment and experience in revising the prior distribution of the optimal spacing. The screen commander must consider first, how BT information is processed to yield line efficiency of his screen and then, how the optimal screening rule applies. In light of these types of considerations we elicit his orderly opinion in the form of the conditional probability distribution which describes the likelihood a value, say sample value m_s, of optimal spacing will be ordered from the results of a BT drop given the value of the optimal spacing s_o. That is, we allow him to evaluate his doctrinal procedures and express the possibility of a deviation of the sample value of spacing from the optimal value of spacing in terms of the likelihood distribution $LK(m_s | s_o)$. For the same reasons that heuristically lead to the assumption of a normally distributed prior, we assume that the normal distribution represents the elicited opinion of the screen commander about the likelihood a sample value of spacing will result from new BT information given that he considers some particular value for the optimal spacing. We can symbolize this subjective probability distribution as

$$LK(m_s | s_o) = N(s_o, V) \tag{8-9}$$

where $LK(m_s|s_o)$ is the conditional probability density function of m_s and $N(s_o,V)$ denotes a normal probability density function with mean s_o and variance V.

Operationally then, a sample of BT information is evaluated by assigning to our PIP system its corresponding probability from the judgmental or likelihood distribution. Using our PIP system we combine this information with previous knowledge to obtain a revised description of the uncertainty about the optimal spacing. Before investigating how to do this we need to pause and reflect on the relationship between the variance V and the prior variance V_{pr}.

Assurance Ratio

Recall that we have elected to quantify uncertainty about a random variable as the reciprocal of the variance. We will define the assurance ratio c as the ratio of the assessed information I_{pr} of the prior uncertainty to the assessed information I_{LK} of the likelihood uncertainty. That is,

$$c = \frac{I_{pr}}{I_{LK}} \tag{8-10}$$

or,

$$c = \frac{V}{V_{pr}} \tag{8-11}$$

Thus, if the screen commander feels his past experience provides relatively little information about the value of the optimal spacing as compared to that information provided by processing current data from a BT drop, then c will be small. Hence c may allow the screen

commander to measure his prior assurance about the optimal spacing. For example, if he is very certain about the optimal value of spacing, that is, V_{pr} will be small, then the value of c will be large. We shall find the assurance ratio to be a convenient quantity for later discussions.

Posterior Distribution

The posterior distribution of the optimal spacing, given sample information, is the output of our probabilistic information processing system. As we will see later, the posterior distribution, denoted $PO(s_o|m_s)$, is the distribution the screen commander will use when choosing a value for the optimal spacing after a BT drop has been made. Using the logic of Bayes theorem to combine the uncertainty expressed by the likelihood distribution with that of the prior distribution, the posterior distribution of the optimal spacing is obtained from the relationship

$$PO(s_o|m_s) = \frac{LK(m_s|s_o)\ PR(s_o)}{\int_{s_o} LK(m_s|s_o)\ PR(s_o)\ ds_o} \tag{8-12}$$

We will now use the mathematical property that normally distributed random variables form a conjugate family. That is, a normally distributed prior combined via Bayes theorem with a normally distributed likelihood will yield a posterior distribution that is also normally distributed. Using the development by Morris (20-73) it can be shown that the posterior probability density function can

symbolized as

$$PO(s_o \mid m_s) = N(m_{po}, V_{po}) \tag{8-13}$$

where the

$$\text{posterior mean} = m_{po} = \frac{cm_{pr} + m_s}{c + 1} \tag{8-14}$$

and the

$$\text{posterior variance} = V_{po} = \frac{V}{c+1} \tag{8-15}$$

Now we examine the expression for the posterior mean and substitute for c the measures of information that form the assurance ratio. We obtain

$$m_{po} = \frac{(\frac{I_{pr}}{I_{LK}})}{(\frac{I_{pr}}{I_{LK}}) + 1} m_{pr} + \frac{m_s}{(\frac{I_{pr}}{I_{LK}}) + 1} \tag{8-16}$$

where I_{pr} and I_{LK} denote the reciprocal of V_{pr} and V respectively. With some algebraic manipulation we obtain

$$m_{po} = (\frac{1}{I_{pr} + I_{LK}}) \left[I_{pr}\, m_{pr} + I_{LK}\, m_s \right] \tag{8-17}$$

Thus, the mean of the posterior destribution of optimal spacing is simply a weighted average of the prior mean and the sample value. Further, we see that the prior mean and sample value are weighted by the relative amount of information that the screen commander feels each source of information should warrant. Hence, as his relative uncertainty becomes greater for one of the inputs to the PIP system,

screen commander's revised opinion reflects this change and permits the other input to become more influential.

From expression (8-15) for the posterior variance, we can take the reciprocal of both sides of this equation to obtain an expression for the measure of information, I_{po}, contained in the posterior distribution. That is,

$$V_{po} = \frac{V}{c+1}$$

or

$$I_{po} = \frac{1}{V}(c+1) \tag{8-18}$$

Substituting information measures for V and c we obtain that

$$I_{po} = I_{pr} + I_{LK} \tag{8-19}$$

Or, in other words, we obtain the interesting result that the amount of information expressed by the screen commander's posterior distribution is the sum of the information quantified by his prior distribution and the likelihood distribution. Thus, in light of our measure of information there is no loss of information when a PIP system is used to process elicited orderly opinions of the screen commander.

Rewriting the expression for posterior variance V_{po} as

$$V_{po} = \frac{V_{pr}}{1 + \frac{1}{c}} \tag{8-20}$$

we can see that the maximum value of the posterior variance V_{po} is the prior variance V_{pr}. Forming the ratio of prior variance and the

posterior variance as

$$\frac{V_{pr}}{V_{po}} = 1 + \frac{1}{c} \tag{8-21}$$

we can observe that the rate the ratio diverges from unity is determined by the slope 1/c. Thus, the smaller the assurance ratio, the more rapidly the screen commander will feel more certain about the output of his PIP system. That is, if the screen commander is more willing to be influenced by sample BT information than by his prior knowledge, then the faster his posterior relative uncertainty will deviate from the prior relative uncertainty.

So much now for the distributions that will be used to find the screen commander's best choice of the screen spacing. We must now develop a distribution that will be required to tackle the question of when to take a BT drop.

Preposterior Distribution

The preposterior distribution is defined as the prior probability distribution of the mean of the posterior probability density function of the optimal spacing, denoted $PR(m_{po})$. Even though the preposterior distribution is a prior probability function, it does not need to be elicited directly from the screen commander. Instead, as will soon show, the preposterior distribution will follow from the probability density functions already elicited, the prior distribution of the optimal spacing and the likelihood distribution of the sample value of optimal spacing. The key to this development is to realize

that the sample value m_s can be treated only as a random variable before sample information is obtained. Because the preposterior distribution is used to make decisions to determine when the sample of BT information will be taken, we see that the sample value of optimal spacing must be handled as a random variable. Using the theorem of total probability we can write the prior expected value, say $E_{pr}(m_s)$, of the sample value of optimal spacing

$$E_{pr}(m_s) = \int_{s_o} E(m_s | s_o)\, PR(s_o)\, ds_o \qquad (8\text{-}22)$$

or

$$E_{pr}(m_s) = \int_{s_o} s_o\, PR(s_o)\, ds_o \qquad (8\text{-}23)$$

It follows immediately that

$$E_{pr}(m_s) = m_{pr} \qquad (8\text{-}24)$$

The prior variance, $V_{pr}(m_s)$, of the sample value of optimal spacing is given by the following development

$$\begin{aligned} V_{pr}(m_s) &= E(m_s - m_{pr})^2 \\ &= E\left[(m_s - s_o) + (s_o - m_{pr})\right]^2 \\ &= E\,(m_s - s_o)^2 + E\,(s_o - m_{pr})^2 \\ &= V + V_{pr} \end{aligned} \qquad (8\text{-}25)$$

We will use Morris' (20-76) development of the prior distribution of m_{po} for the case when all distributions are normal. Recall that we have

$$m_{po} = \frac{c\, m_{pr} + m_s}{c + 1} \tag{8-26}$$

Taking the expected value of m_{po} we obtain

$$E_{pr}(m_{po}) = \frac{c\, m_{pr} + E_{pr}(m_s)}{c + 1} \tag{8-27}$$

or

$$E_{pr}(m_{po}) = m_{pr}. \tag{8-28}$$

Again using Equation (8-26) we obtain the prior variance of m_{po} as follows:

$$V_{pr}(m_{po}) = \left(\frac{1}{c + 1}\right)^2 V_{pr}(m_s) \tag{8-29}$$

or, using Equation (8-25), we have

$$V_{pr}(m_{po}) = \left(\frac{1}{c + 1}\right) V_{pr} \tag{8-30}$$

With some algebra we can show that

$$V_{pr}(m_{po}) = V_{pr} - V_{po} \quad . \tag{8-31}$$

We will see later the importance of the prior distribution of the posterior mean. However, let us rewrite Equation 8-30 after

substituting for c in Equation 8-30 as follows:

$$V_{pr}(m_{po}) = \frac{V_{pr}^2}{V + V_{pr}} \tag{8-32}$$

This expression allows us to note that as the variance of the sample values of optimal spacing increases, that $V_{pr}(m_{po})$ decreases. This says simply that our prior uncertainty about m_{po} increases as our uncertainty about the value of optimal spacing from the same information decreases. Or, we see that the more willing the screen commander is to let sample information influence the decision the less certain he is about his prior opinion about the posterior mean.

Summary of Distributions

Before we proceed to the informational phase of the analysis, we summarize the distributions that we have been discussing in the following list.

SUMMARY OF DISTRIBUTIONS

1. $PR(p|s)$ The prior distribution of the probability of submarine hitting given it fires just outside a screen with a given spacing

$$PR(p|s) = F(m_H, V_H)$$

where $m_H = 1 - ks.$

2. $PR(q|s)$ The prior distribution of a submarine's undetected penetration of a screen with a given spacing.

$$PR(q|s) = F(m_d, V_d)$$

3. $PR(s_o)$ The prior distribution of the optimal spacing

$$PR(s_o) = N(m_{pr}, V_{pr}).$$

4. $LK(m_s|s_o)$ The likelihood distribution of a sample value of optimal spacing given the value of the optimal spacing.

$$LK(m_s|s_o) = N(s_o, V).$$

5. $PO(s_o|m_s)$ The posterior distribution of the optimal spacing after the receipt of sample information

$$PO(s_o|m_s) = N(m_{po}, V_{po})$$

where

$$m_{po} = \frac{cm_{pr} + m_s}{c + 1}$$

and

$$V_{po} = \frac{cV_{pr}}{c+1}$$

6. $PR(m_{po})$ The prior distribution of the mean of the posterior distribution of the optimal spacing

$$PR(m_{po}) = N(m_{pr}, V_{pr} - V_{po})$$

CHAPTER IX

THE ECONOMIC ANALYSIS

The Economic Impact of Uncertainty

The significance of understanding the economic implication of uncertainty in decision making has already been pointed out. Our task in this chapter is to make this concept operational for the screen commander as he attempts to solve the screen placement problem. We are already aware that the cost of BT information and the cost of an error in spacing is measured by an increase in the probability of submarine hitting. We shall use this probability of submarine hitting as our economic measure to analyze what spacing the screen commander should order so that he may minimize this probability. Further, we will investigate when it will be beneficial for the screen commander to order a BT drop. Intuitively a BT drop should be conducted only when the return is greater than the cost. To obtain insight to these two problems of the screen commander is the goal of the economic analysis.

Prior Expected Payoff

Let us recall that we have introduced the prior distribution of the probability of submarine hitting given it fires just outside a screen with a given spacing, $(p|s)$. We define the prior expected payoff for a given spacing, say, PREP, as the expected value of $(p|s)$. Thus,

$$PREP = \int_p p \; PR(p|s) \; dp \qquad (9\text{-}1)$$

or,

$$PREP = m_H \tag{9-2}$$

or, as we have indicated in our previous discussion,

$$PREP = 1 - ks \tag{9-3}$$

We have already recognized that the screen commander should choose the value of spacing for which the probability of submarine hitting from outside the screen was equal to the probability of successful penetration of this same screen. However, we have also recognized that the screen commander may be uncertain about what spacing corresponds to the intersection of the curves that represent these probabilities. Thus, a reasonable interpretation of the optimal screening rule seems to be to choose that spacing for which the expected value of the probability of submarine hitting from outside the screen is equal to the expected value of successful penetration of this same screen. Or, to symbolize this statement, the screen commander would order the value of spacing, say $\tilde{s}$, for which

$$(m_H|\tilde{s}) = (m_d|\tilde{s}) \tag{9-4}$$

Thus, when the screen commander intuitively integrates his uncertainties in light of Equation (9-4), we can write his prior expected payoff for the chosen spacing $\tilde{s}$, say $(PREP|\tilde{s})$, as follows:

$$PREP|\tilde{s} = 1-k\tilde{s} \tag{9-5}$$

Because we have designed our probabilistic information processing system to revise the screen commander's personal probability about the optimal spacing, we will now show what value of spacing, based on the prior distribution of optimal spacing, would correspond to the particular value of spacing $\tilde{s}$ in order to achieve the minimum probability of submarine hitting. If the screen commander were to choose the optimal spacing as his ordered spacing, then he would indeed be ordering the spacing such that the minimum probability of submarine hitting, say MP, would result. In this case, we can write

$$MP = 1-ks_o \tag{9-6}$$

Recalling the gain function $g(s_o)$, which seemed to suggest itself from the observation of minimizing the deviation between the estimated successful penetration function and the actual penetration function, when we were treating our screen placement problem deterministically, we can write again

$$g(s_o) = \left|(1 - LE_{EST}|s) - (1 - LE_{ACT}|s_o)\right|. \tag{9-7}$$

In view of our probabilistic interpretation of the optimal screening rule, we have for the case when the ordered spacing s is the particular spacing $\tilde{s}$ that

$$(1 - LE_{EST}|\tilde{s}) = (m_d|\tilde{s}) \tag{9-8}$$

but $$(m_d|\tilde{s}) = (m_H|\tilde{s})$$

thus,

$$(1 - LE_{EST} \mid \tilde{s}) = 1 - k\tilde{s}. \tag{9-9}$$

From the optimal screening rule, we have

$$(1 - LE_{ACT} | s_o) = 1 - ks_o \tag{9-10}$$

Thus, substituting Equations (9-9) and (9-10) into the equation for our gain function, we have

$$g(s_o) = |(1 - k\tilde{s}) - (1 - ks_o)| \tag{9-11}$$

or

$$g(s_o) = |k(s_o - \tilde{s})| \tag{9-12}$$

Thus, we can interpret the gain function as yielding the differences between the prior expected payoff for the particular choice of spacing $\tilde{s}$ and the probability of submarine hitting, given the optimal spacing. However, because the optimal spacing is a random variable our expression for the gain function can only be viewed as conceptual because the value of optimal spacing is not known, for if it were, then the screen commander would choose as his ordered spacing the optimal spacing. In order to make the gain function operational, we will incorporate with it our probabilistic information processing system that has been designed to handle the screen commander's uncertainty associated with the screen placement problem. Thus, we now elect to express the expected value of the deviation between the minimum probability of submarine hitting and the prior expected payoff for the particular case

when the spacing is $\tilde{s}$. This expected value we denoted as $E\left[g(s_o)\right]$ and is as follows:

$$E\left[g(s_o)\right] = \int_{s_o} g(s_o)\, P(s_o)ds_o \qquad (9\text{-}13)$$

or

$$E\left[g(s_o)\right] = \int_{s_o=0} \left|k(s_o-\tilde{s})\right| P(s_o)ds_o \qquad (9\text{-}14)$$

Using the classical optimization techniques of calculus, we shall now solve for that value of spacing to order, say $\approx{s}$, which minimizes the expected gain, $E\left[g(s_o)\right]$. Rewriting Equation (9-14), we have

$$E\left[g(s_o)\right] = \int_{s_o=0}^{\tilde{s}} k(\tilde{s}-s_o)\, P(s_o)ds_o + \int_{s_o=\tilde{s}}^{\infty} k(s_o-\tilde{s})\, P(s_o)ds_o \qquad (9\text{-}15)$$

Now, taking the first partial derivative with respect to the particular ordered spacing $\tilde{s}$ of the expected gain, we have

$$\frac{\delta E\left[g(s_o)\right]}{\delta \tilde{s}} = k \int_{s_o=0}^{\tilde{s}} P(s_o)ds_o - k \int_{s_o=\tilde{s}}^{\infty} P(s_o)ds_o \qquad (9\text{-}16)$$

Setting this first partial derivative equal to zero, we obtain

$$\int_{s_o=0}^{\tilde{s}} P(s_o)\, ds_o = \int_{s_o=\tilde{s}}^{\infty} P(s_o)ds_o \qquad (9\text{-}17)$$

Thus, the candidate for the value of spacing $\approx{s}$ which satisfies Equation (9-17) is the median of the probability density function for the

optimal spacing or for a symmetrical distribution, it is the expected value of s_o. That is, when the following expression is satisfied, the expected gain may be minimized:

$$\approx{s} = E(s_o) \tag{9-18}$$

To determine the nature of the stationary point at $\approx{s} = E(s_o)$, we make the conventional test on the sign of the second partial derivative with respect to $\tilde{s}$ of the expected gain function $E\left[g(s_o)\right]$. That is,

$$\frac{\delta^2 E\left[g(s_o)\right]}{\delta \tilde{s}^2} = k\ P(\tilde{s}) - \left[-k\ P(\tilde{s})\right] \tag{9-19}$$

or

$$\frac{\delta^2 E\left[g(s_o)\right]}{\delta \tilde{s}^2} = +\ 2k\ P(\tilde{s}) \tag{9-20}$$

Thus, the sign of the second partial derivative evaluated at the point where $\tilde{s}$ is equal to the mean or median of the distribution is positive. Thus the expected gain function is minimized, if the value of spacing is chosen to be the median of the probability density function of s_o, or for the case when the probability density function of optimal spacing is symmetrical, the best value of spacing is the expected value of optimal spacing.

The Best Prior Act and the Best Posterior Act

We must interpret the results of the last section as indicating that the screen commander's best act or choice of spacing to order is that spacing which corresponds to the median of the probability distribution of the optimal spacing. Or, for the normally distributed case, as we have assumed in our analysis, his best act is the mean of the distribution. That is, the expected value of optimal spacing minimizes the expected gain function and hence, is the best choice of spacing the screen commander can order to obtain a minimum probability of submarine hitting. Thus, the best prior act results in the following prior expected payoff:

$$PREP|\tilde{\tilde{s}} = 1 - k\, m_{pr} \qquad (9\text{-}21)$$

The time of an act by the screen commander has been indicated by whether or not it takes place before or after he obtains BT information. Fortunately, the result of the last section does not depend on when the screen commander acts, if we observe that for the prior expected payoff given the particular spacing $\tilde{s}$, we have a corresponding posterior expected payoff given the particular spacing $\tilde{s}$, say $POEP|\tilde{s}$ Implicitly, we are assuming the screen commander obtains BT information, integrates this information and is ready again to assess his opinions. consequently, the same argument will lead us to assign a value to $POEP|\tilde{s}$ by the same method we assigned a value to $PREP|\tilde{s}$. That is, we have

$$POEP|\tilde{s} = 1 - k\tilde{s} \qquad (9\text{-}22)$$

Thus, the best posterior act results in the following posterior expected payoff:

$$\mathrm{POEP}\big|_{\tilde{\tilde{s}}} = 1 - k\, m_{po} \tag{9-23}$$

Thus, we can say that if the screen commander must choose a spacing to order before additional information can be obtained, the prior distribution of optimal spacing is the applicable distribution. Hence, the best prior act is the prior mean, m_{pr}. When the opportunity to obtain information is taken, then the best posterior act that the screen commander can choose is m_{po}, the expected value of the posterior distribution of the optimal spacing.

A question which immediately suggests itself is, do the above results make sense? When we discussed the eliciting of the prior distribution, we asked for the most likely value of optimal spacing and designated this as the prior mean, m_{pr}. Also, when we discussed the integration of the prior distributions, $PR(p|s)$ and $PR(q|s)$, which caused the screen commander to reflect on the options of the submarine commander, we recognized when the means of these distributions were nearly equal, that the given spacing for these distributions would correspond nearly to the mean m_{pr}. Thus, to find that the mean value of optimal spacing is the best act to minimize the deviation between the minimum probability of submarine hitting and the prior expected payoff for the particular case when the optimal screening rule is being used, comes as no surprise and hence is more than appealing. That is, the result makes sense.

To summarize, with the help of a PIP system, the screen commander can order the best act or mean value of optimal spacing as the spacing for his screening unit, to use both before and after BT information is available. The significant result is that we are not required to elicit the distributions of $(p|s)$ and $(q|s)$ in order to choose the best acts. Thus, even though the distributions of $(p|s)$ and $(q|s)$ have allowed us to complete our investigation into what spacing to order, they can be regarded as a conceptual means to an end. Now, however, to complete our solution to the screen placement problem, we must address our attention to the question of when to obtain BT information.

The Measure of the Cost of Information

Recall that the probability of submarine hitting from outside the screen, denoted p, can be represented as a linear function of spacing as follows:

$$p = 1 - k\, s \qquad (9\text{-}24)$$

where k is the slope that is used to represent the potential of the submarine's weapons against a screen with a given number of screening units. The effectiveness of the submarine's torpedoes is seen to fall off quickly as the spacing becomes larger, if the value of the slope k is large. We have explicitly regarded k as being part of the screen commander's estimate and as such, have viewed k as being a known constant in the screen placement problem.

We are already aware from Chapter IV that when the screen commander detaches a screening unit to obtain BT information, it incurs a decrease in his efficiency because he must increase the spacing between the remaining units in order to fill the gap left in the screen by the departing screening unit. Let s_1 and s_2 represent the ordered spacing before BT information is sought and the increased spacing while the drop is being made, respectively. Also, let k_1 and k_2 be the known slope for the case where the number of screening units are, as before and during the BT drop, respectively. Because the screen commander will be attempting to follow the optimal screening rule, we have the probability of submarine hitting, say p_1 and p_2 for the two cases mentioned above, respectively as follows:

$$p_1 = 1 - k_1 s_1 \qquad (9\text{-}25)$$

and,

$$p_2 = 1 - k_2 s_2 \qquad (9\text{-}26)$$

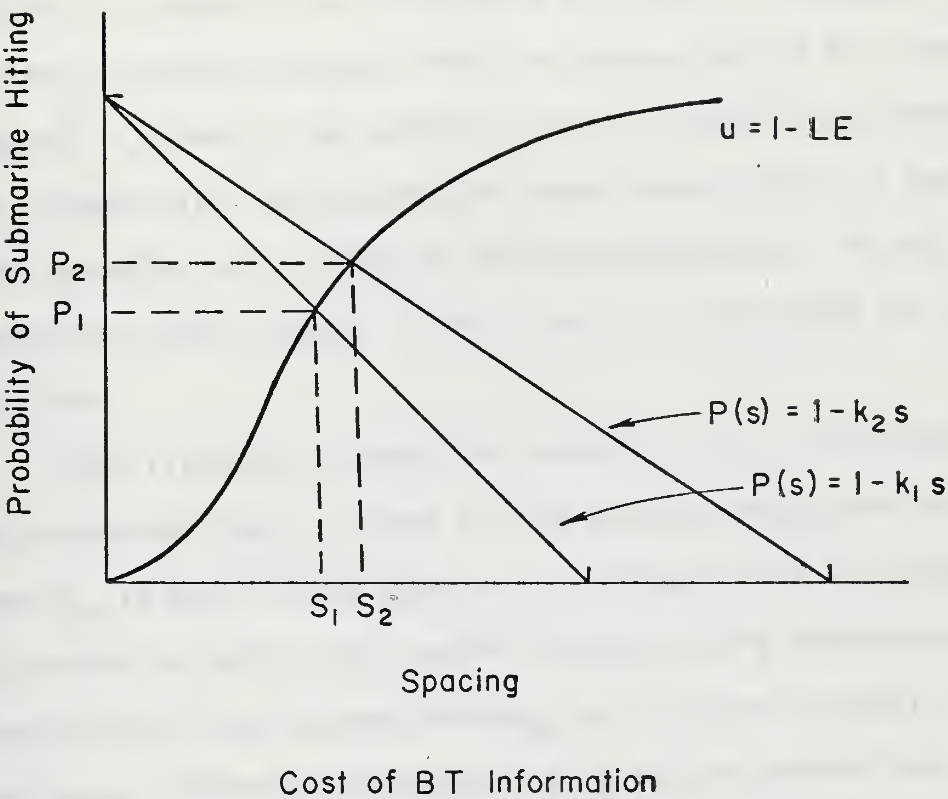
The difference between p_2 and p_1 represents the increase in probability of submarine hitting when the screen commander elects to obtain BT information. We can write this cost of information, denoted C_I, as follows

$$C_I = p_2 - p_1 \qquad (9\text{-}27)$$

or

$$C_I = k_1 s_1 - k_2 s_2 \qquad (9\text{-}28)$$

This discussion is summarized in Figure 19.



Probability of Submarine Hitting
u = 1 - LE
P2
P1
P(s) = 1 - k2 s
P(s) = 1 - k1 s
S1 S2
Spacing
Cost of BT Information

FIGURE 19

Loss Given Sample Information

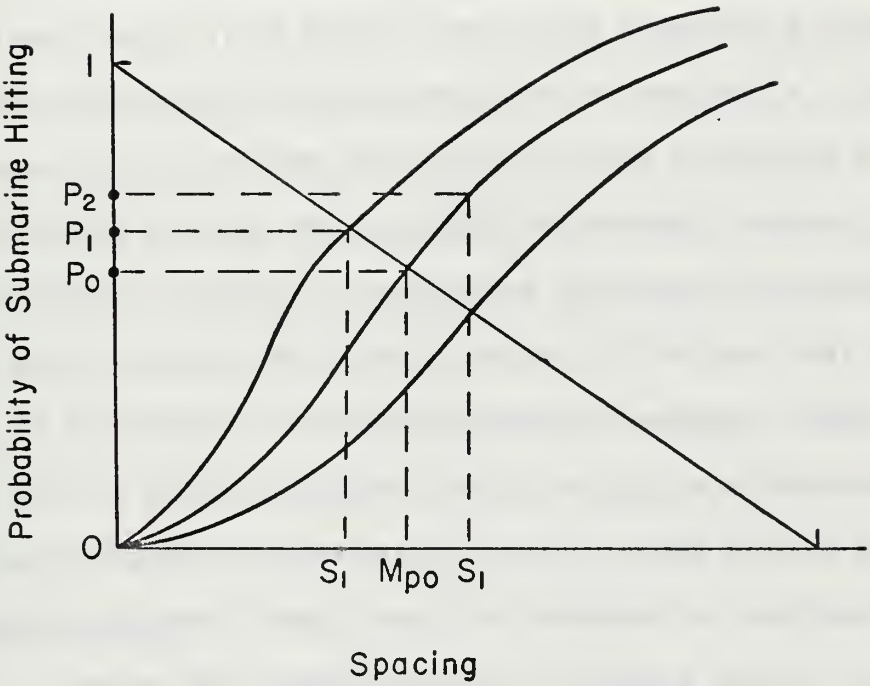
We have already shown that if the screen commander is to decide on a spacing before sample or BT information is obtained, then his best prior act is to order the spacing that corresponds to his prior mean of the optimal spacing. We now use the loss function developed in Chapter VI to investigate the loss, or increase in probability of submarine hitting, that is incurred before the present spacing, s_1, used by the screening units is changed in accordance with our probabilistic information processing system--that is, before the best posterior act is taken by the screen commander. We will need to examine two cases similar to those when we investigated the loss function.

The first case is when the posterior mean of the optimal spacing is greater than, or equal to, the present spacing and the other, when m_{po} is less than or equal to s_1. Using Figure 20, which is Figure 15 revised to reflect the current situation being investigated, we can represent the loss incurred when m_{po} is less than or equal to s_1 as the difference between the probability of submarine hitting that corresponds to his present spacing, or symbolically

$$p_1 = 1 - k\ s_1 \qquad (9\text{-}29)$$

and that probability of submarine hitting that is the posterior expected payoff, or,

$$p_o = 1 - k\ m_{po} \qquad (9\text{-}30)$$



Loss Given Sample Information

FIGURE 20

The second case can be described by the same difference as that above but in this case the probability of submarine hitting that corresponds to his present spacing s_1 is represented by the following:

$$p_2 = 1 - LE_{EST}|s_1 \qquad (9\text{-}31)$$

Let us pause here to reflect on the quantity $(1 - LE_{EST}|s_1)$, which from Chapter VI we know is the screen commander's estimated successful penetration function evaluated at spacing, s_1. Recall in our discussion of the line efficiency function in Chapter III that we assumed the LE function was monotomic decreasing. Because the successful penetration function, or undetected penetration function, say U, is one minus the line efficiency function, it follows that the function U must be a monotonic increasing function of spacing. Thus, what is called for, in order to obtain a value for p_2, is a procedure for estimating an undetected penetration function which has the monotonic increasing property. Thus, when U is evaluated at the best posterior act, m_{po}, and at the present spacing, it follows when m_{po} is less than or equal to s_1, that is,

$$U(s = m_{po}) \leq U(s = s_1) \qquad (9\text{-}32)$$

Let us note now, that an estimated U function must also be considered a function of m_{po}, in order to insure the monotonic property holds. Hence, we shall denote the quantity $(1 - LE_{EST}|s_1)$ as $U(m_{po}, s_1)$. Further, we must note that the undetected penetration function evaluated

as m_{po}, corresponds to the intersection of the U function and the linear function of probability of submarine hitting, or,

$$U(m_{po}) = 1 - k\, m_{po} \tag{9-33}$$

We will postpone a specific discussion of how the quantity $U(m_{po}, s_1)$ can be obtained until we can operationally analyze our model. For now, we will assume that a procedure for obtaining $U(m_{po}, s_1)$ is available to the screen commander as he reflects on the losses he may incur if his spacing between screening units is in error.

We can now summarize the loss given sample information, $L|SI$, that results if the present spacing is not changed to the best posterior act, m_{po}. Thus we have

$$L|SI = \begin{cases} p_1 - p_o & m_{po} \geq s_1 \\ p_2 - p_o & m_{po} \leq s_1 \end{cases} \tag{9-34}$$

or, we can write

$$L|SI = \begin{cases} k(m_{po} - s_1) & m_{po} \geq s_1 \\ U(m_{po}, s_1) - (1 - km_{po}) & m_{po} \leq s_1 \end{cases} \tag{9-35}$$

Because the loss given sample information identifies to the screen commander the increase in probability of submarine hitting, that indeed will not be incurred when the best posterior act is taken, we can view this change in probability of submarine hitting as the value of sample information, say VSI. Thus, we can write

$$\mathrm{VSI} = \begin{cases} k(m_{po} - s_1) & m_{po} \geq s_1 \\ U(m_{po}, s_1) - (1 - k\, m_{po}) & m_{po} \leq s_1. \end{cases} \qquad (9\text{-}36)$$

<u>Expected Value of Sample Information</u>

When addressing the question of whether or not to obtain information in order to revise the optimal spacing distribution, the distribution that becomes significant is the preposterior distribution, because it reflects the screen commander's uncertainty about how the results of a future BT drop will affect the best posterior act, m_{po}. Because, as we have shown in the development of the preposterior distribution, the posterior mean m_{po} can be regarded as a random variable before BT information is obtained, we can find the expected value of the loss given sample information developed in the last section. Thus, using the preposterior distribution of m_{po}, we can find the expected value of sample information, say EVSI, as follows:

$$\mathrm{EVSI} = \int_{m_{po}=-\infty}^{\infty} (\mathrm{VSI})\ \mathrm{PR}(m_{po})\ d\, m_{po} \qquad (9\text{-}37)$$

Or, substituting Equation (9-36) into (9-37), we have

$$\mathrm{EVSI} = \int_{m_{po}=s_1}^{\infty} k(m_{po} - s_1)\ \mathrm{PR}(m_{po})\ d\, m_{po}$$

$$+ \int_{m_{po}=-\infty}^{s_1} \left[U(m_{po}, s_1) - (1 - k\, m_{po}) \right] PR(m_{po}) d\, m_{po} \qquad (9\text{-}38)$$

For discussion purposes, let

$$EVSI = I_1 + I_2, \qquad (9\text{-}39)$$

where I_1 and I_2 stand for the first and second integrals in Equation (9-38) respectively. Let us notice that because the limits of integration on I_1 is from s_1 to infinity, that the quantity $(m_{po} - s_1)$ will always be greater than, or equal to zero. Because $PR(m_{po})$ represents a non-negative number between zero and one, we have that the value of I_1 will be greater than or equal to zero. Using Equation (9-33), we can write that

$$I_2 = \int_{m_{po}=-\infty}^{s_1} \left[U(m_{po}, s_1) - U(m_{po}) \right] PR(m_{po})\, d\, m_{po} \qquad (9\text{-}40)$$

From the monotonic assumption, we see that over the limits of integration of I_2, the quantity $U(m_{po}, s_1) - U(m_{po})$, is greater than, or equal to zero. Hence, it follows that the value of I_2 will be greater than or equal to zero. Combining these observations about I_1 and I_2, we have that the sum of these two integrals, or the expected value of sample information, EVSI, is greater than or equal to zero. Thus,

$$EVSI \geq 0 . \qquad (9\text{-}41)$$

Expected Net Gain of Sample Information

As usual in Bayesian economic analyses, the difference between the expected value of sample information and the cost of obtaining information will be called the expected net gain from sample information, say ENGSI. We can represent this statement as follows:

$$ENGSI = EVSI - C_I \tag{9-42}$$

The expected net gain from sample information provides the screen commander with a measurable quantity that he can use when deciding if a BT drop should be conducted. A decision rule which suggests itself is, that if ENGSI is positive, then the screen commander should order a BT drop and if the ENGSI is non-positive, he should wait. Or, in other words, if the anticipated loss derived from BT information is greater than the increase in probability of submarine hitting that results when a BT drop is being made, then the information should be obtained. Substituting in Equation (9-42), we have

$$\begin{aligned} ENGSI = k \int_{m_{po}=s_1}^{\infty} & (m_{po} - s_1)\, PR(m_{po})\, dm_{po} \\ & + \int_{m_{po}=-\infty}^{s_1} \left[U(m_{po}, s_1) - (1 - km_{po}) \right] PR(m_{po}) dm_{po} \\ & - (ks_1 - k_2 s_2) \end{aligned} \tag{9-43}$$

Thus, we have the result that was the final part of the goal of our economic analysis, a quantitative method by which the screen commander can decide when to order a BT drop.

However, now that we have developed a Bayesian Decision model, we must now complete our investigation in the sense that we must make the model operational, so that the results can be examined in order to see if they make sense to the screen commander. This will be the topic of the next chapter.

CHAPTER X

THE OPERATIONAL ANALYSIS

Preliminary Remarks

We have already established when the screen commander should seek to obtain BT information, that being when the expected net gain of sample information is positive. We are now faced with the task of interpreting this result to see if it is appealing or makes sense to the screen commander in light of our present model. Or, in other words, can we establish a decision rule in terms of quantities discussed in the development of the model which will permit us to realize in a straight forward manner when ENGSI may be positive. We are also interested in what effect changes in these quantities identified in our model will have on ENGSI.

Because the difference of the expected value of sample information and cost of information provides us with ENGSI and that the cost of information is rather simply determined in terms of the quantities of our model of the screen placement problem, we will focus our attention on the expected value of sample information. This, in turn, requires first a discussion of how line efficiency functions, or undetected penetration functions can be generated. We will then use these functions, in the general and specific cases, to examine EVSI.

Estimation of Line Efficiency/Undetected Penetration Functions

We have assumed in the last chapter that a procedure is available for obtaining or estimating line efficiency function, or undetected penetration function, so that the screen commander indeed can estimate the losses he may incur if his spacing is in error, especially when the best posterior act is less than the present spacing. From Chapter III, we have the idea that a family of curves will represent the LE or U function that is appropriate to use to estimate the losses. Since a family of curves is usually represented by a general function with a parameter that can vary to produce a specific function, we need to investigate a procedure for evaluating the parameter of a family of candidate functions. Because we use these estimated LE or U functions to determine if BT information should be obtained, the posterior best act, m_{po}, may play a role in determining the value of this parameter. The use of m_{po} to find the value of the parameter will also guarantee, in the case when m_{po} is less than or equal to the present spacing that the value of U for a given m_{po} will be less than or equal to U, evaluated at the present spacing s_1, if the family of curves is monotonic increasing as is required in our model.

A procedure to find the value of the parameter of a family of curves that seems to suggest itself is, that the value of the linear function of isoprobability of submarine hitting for a given m_{po} will be set equal to the value of the undetected penetration function for the same given m_{po}. In this way, the undetected penetration function, corresponding to any of the screen commander's estimate of the

posterior best act, can be generated. As examples of such a procedure, we will now develop the undetected penetration function for each of the two general representations of the line efficiency functions, given in Chapter III.

In the first case, recall from Equation (3-2) that

$$LE = e^{-\lambda s} \tag{10-1}$$

or,

$$1 - LE = 1 - e^{-\lambda s} . \tag{10-2}$$

Following the suggested procedure for some m_{po}, we have

$$1 - k\, m_{po} = 1 - e^{-\lambda m_{po}} \tag{10-3}$$

or

$$k\, m_{po} = e^{-\lambda m_{po}} \tag{10-4}$$

Solving for λ , the parameter of the family of curves, in order to obtain a specific LE or U function, we get

$$\lambda = \frac{\ln\left(\frac{1}{k\, m_{po}}\right)}{m_{po}} \tag{10-5}$$

or

$$\lambda = \ln(k\, m_{po})^{-\frac{1}{m_{po}}} \tag{10-6}$$

Thus, we have

$$LE = e^{-\ln(k\, m_{po})^{-\frac{1}{m_{po}}} \cdot s} \tag{10-7}$$

or

$$LE = (k\, m_{po})^{\frac{s}{m_{po}}} \tag{10-8}$$

Similarly, for the second case we use Equation (3-4) and obtain

$$\ln m_{po} = e^{-\lambda m_{po}^{2}}, \qquad (10\text{-}9)$$

hence,

$$\lambda = \frac{\ln\left(\frac{1}{k\, m_{po}}\right)}{m_{po}^{2}} \qquad (10\text{-}10)$$

or,

$$LE = (k\, m_{po})^{\left(\frac{s}{m_{po}}\right)^{2}} \qquad (10\text{-}11)$$

Thus we have the generation of an estimated line efficiency function that also has the monotonic property for the two cases suggested in Chapter III.

Investigation of Expected Value of Sample Information

As indicated, the expected value of sample information plays an important role in deciding when to sample for BT information. Our purpose is now to investigate this quantity in order to provide some possible insight to the screen placement problem. From Equation (9-38) we have the expression for the expected value of sample information as follows:

$$EVSI = \int_{m_{po}=s_1}^{\infty} k\,(m_{po} - s_1)\, PR(m_{po})\, d\, m_{po} + \int_{m_{po}=-\infty}^{s_1} \left[U(m_{po}, s_1) - (1 - k\, m_{po}) \right] PR(m_{po})\, d\, m_{po} \qquad (10\text{-}12)$$

As before, let

$$EVSI = I_1 + I_2 \qquad (10\text{-}13)$$

We shall now investigate the values of the first and second integrals in the expression for EVSI.

Recalling that the preposterior distribution is as follows:

$$PR(m_{po}) = N(m_{pr}, V\ m_{po}), \qquad (10\text{-}14)$$

we can evaluate the expression for I_1 as a right-hand Linear Normal Loss Integral, which is tabulated in Reference (20). Thus we may write the integral I_1 as follows:

$$I_1 = k\ L_{RH}\ (s_1)\ SD(m_{po}), \qquad (10\text{-}15)$$

where $L_{RH}(s_1)$ is the value from the Linear Normal Loss Table and $SD(m_{po})$ is the standard deviation of the preposterior distribution. Because the particular value of $L_{RH}(s_1)$ depends on whether the expected value of the best posterior act is greater than or less than the present spacing, $L_{RH}(s_1)$ can be calculated using one of the two following cases.

For the case when $E(m_{po})$ is less than or equal to s_1,

$$L_{RH}(s_1) = L_N \left[Zs_1 = \frac{s_1 - E(m_{po})}{SD(m_{po})} \right] \qquad E(m_{po}) \leq s_1 \qquad (10\text{-}16)$$

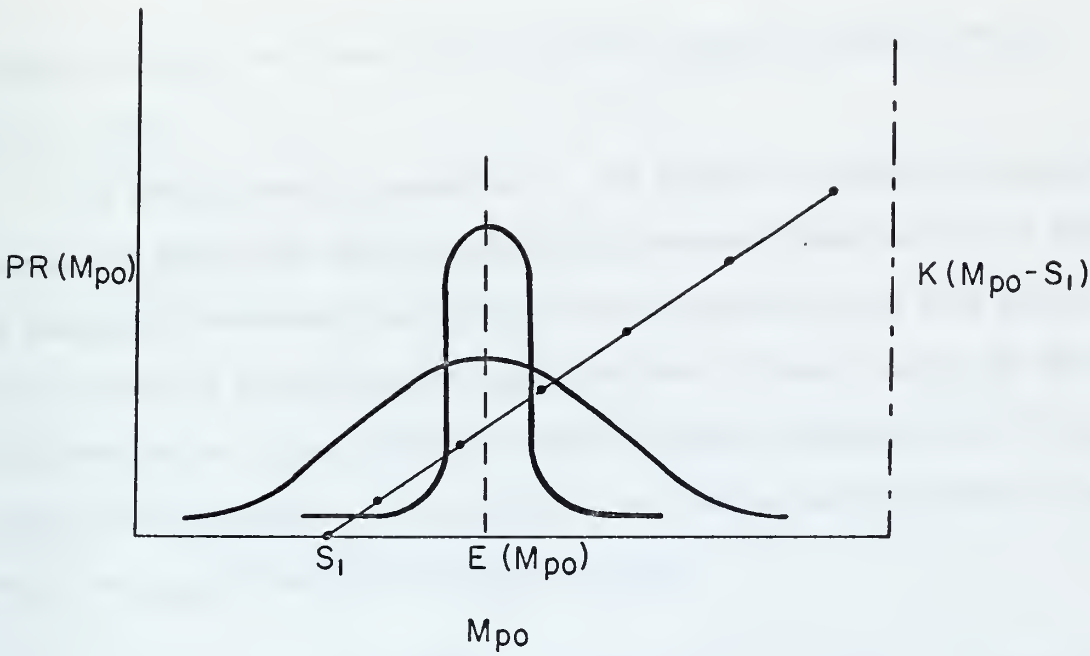
When $E(m_{po})$ is greater than or equal to the present spacing we have

$$L_{RH}(s_1) = L_N\left[Zs_1 = \frac{E(m_{po}) - s_1}{SD(m_{po})}\right] + Zs_1 \qquad E(m_{po}) \geq s_1 \tag{10-17}$$

where

$$Zs_1 = \frac{E(m_{po}) - s_1}{SD(m_{po})} \tag{10-18}$$

Using the property of the Linear Normal Loss Integral, that if Zs_1 is less than Zs_1', then $L_N(Zs_1)$ is greater than $L_N(Zs_1')$, we can see that as the quantity $SD(m_{po})$ increases that Zs_1 becomes smaller, or that the value of $L_N(Zs_1)$ increases. With this result, we again examine the expression for I_1 and immediately note that I_1 is an increasing function of the standard deviation of the preposterior distribution. Using Figure 21 to heuristically relate what has been developed, we note that an interpretation which seems to suggest itself to describe the increase in I_1 as $SD(m_{po})$ increase, is that an increase in $SD(m_{po})$ implies that the larger values of the quantity $k(m_{po}-s_1)$ become more likely. Hence, since our integration, with which we are concerned, yields the summation of the products of the quantity $k(m_{po}-s_1)$ and its associated probability of happening, an increase in I_1 as $SD(m_{po})$ increases seems reasonable.



$$\text{On Computing } I_1 = \int_{M_{po}=S_1}^{+\infty} k\,(M_{po} - S_1)\; PR\,(M_{po})\; d\,M_{po}$$

FIGURE 21

Now, let us focus on the integral I_2 in the expression for expected value of sample information. Let

$$\bar{U}(m_{po}) = U(m_{po}, s_1) - U(m_{po}) \qquad 10\text{-}19)$$

which represents the loss that is incurred when the present spacing is in error.

We shall restrict ourselves to the family of curves as discussed previously which also has a methematical property such that for a family of monotonic increasing functions, which originate at the same point, there exists no intersection between any two different curves of the family. This same point is the origin in Figure 22 which indicates that for a zero spacing between units, the probability of undetected penetration is zero. Thus, for $m_{po}(2)$ greater than $m_{po}(1)$ we have

$$U(m_{po}{}^{(1)}, s_1) - U(m_{po}{}^{(2)}, s_1) < 0. \qquad (10\text{-}20)$$

Therefore, the derivative of the function $U(m_{po}, s_1)$ with respect to m_{po} is less than zero, or

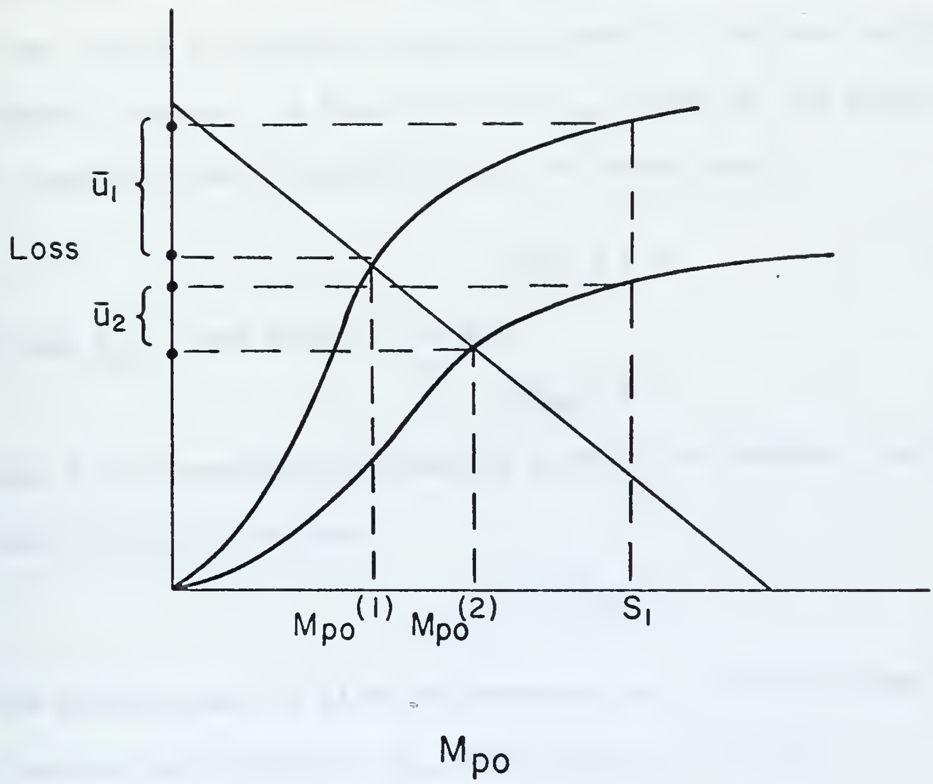
$$\frac{dU(m_{po}, s_1)}{dm_{po}} < 0 \qquad (10\text{-}21)$$

Now, taking the derivative of expression (10-19) for $\bar{U}$ and recalling that

$$U(m_{po}) = 1 - k\, m_{po}$$

we obtain

$$\frac{d\bar{U}}{dm_{po}} = \frac{dU(m_{po}, s_1)}{dm_{po}} + k \qquad (10\text{-}22)$$



Loss as M_{po} Approaches S_1

FIGURE 22

which may be either positive or negative depending on the relationship between the absolute value of expression (10-21) for $dU(m_{po}, s_1)/dm_{po}$ and k. We are comparing the change in the undetected penetration probability given the present spacing s_1 as the best posterior act changes to the change in posterior expected payoff as the best posterior act changes. However, we know that for m_{po}, equal to the present spacing s_1, that the loss incurred is zero, or symbolically

$$\overline{U}(s_1) = 0 \tag{10-23}$$

For any m_{po} less than s_1, we have

$$\overline{U}(m_{po}) > 0 \tag{10-24}$$

since U is a monotonic increasing function of spacing. For any m_{po} greater than s_1, we have

$$\overline{U}(m_{po}) < 0 \tag{10-25}$$

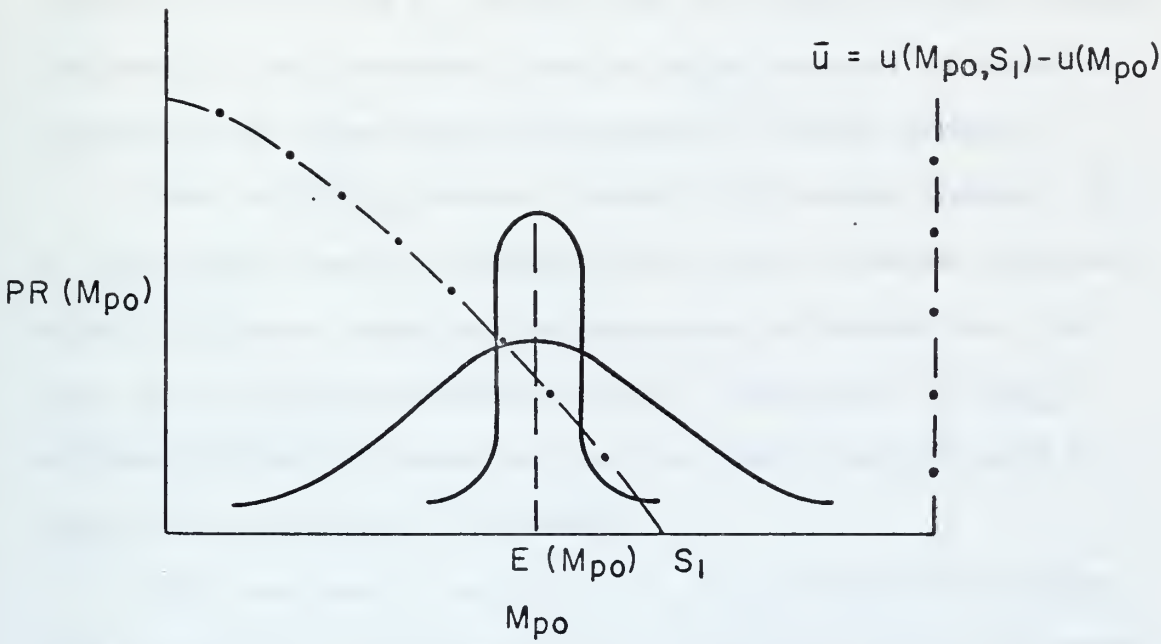
These observations we elect to interpret as indicating that $\overline{U}(m_{po})$ is a decreasing function as m_{po} approaches s_1, or that

$$\frac{d\overline{U}}{dm_{po}} < 0 \qquad m_{po} < s_1 \tag{10-26}$$

Using again

$$PR(m_{po}) = N(m_{pr}, Vm_{po}) \tag{10-27}$$

we can summarize the calculation of the partial expected loss of this decreasing function of m_{po} in Figure 23, which is similar to Figure 21.



On Computing $I_2 = \int_{M_{po}=-\infty}^{S_1} \left[u(M_{po},S_1)-u(M_{po})\right] PR(M_{po})\, dM_{po}$

FIGURE 23

From our interpretation of I_1, a result that is heuristically implied is that as the SD(m_{po}) increases, the value of I_2 will also increase because the greater values of $\bar{U}$ become more probable.

Hence, if we combine the reasonable interpretations of our investigations of I_1 and I_2, we have that the expected value of sample information is an increasing function of the standard deviation or variance of the preposterior distribution of optimal spacing.

Thus, as SD(m_{po}) becomes greater, EVSI becomes greater. Or, in other words, when the reliability the screen commander expresses about his opinions about the best posterior act becomes less, the value of BT information becomes greater. Conversely, as SD(m_{po}) decreases,EVSIwill decrease and the less likely that EVSI will be greater than the cost of information.

This last result indicates that the procedure for deciding when to obtain BT information does not ignore the screen commander's experience, or subjective probability that describes his feelings about the location of the best posterior act, m_{po}. Indeed, we are able to use the screen commander's indication of the reliability of his estimates explicitly in the evaluation of ENGSI, which in turn, suggests when a decision to obtain BT information should be made. This certainly is an improvement over the aspiration level handling of information indicated in our discussion of the PIP system.

Let us investigate EVSI further, by looking at I_2 for the specific U functions that are obtained from the example line efficiency function, that were generated earlier in the chapter. Hence,

substituting (10-8) and (10-11) into the expression for I_2, yields

$$I_2 = \int_{m_{po}=-\infty}^{s_1} \left[k\, m_{po} - (k\, m_{po})^{\frac{-s}{m_{po}}} \right] PR(m_{po})\, dm_{po} \quad (10\text{-}28)$$

and

$$I_2 = \int_{m_{po}=-\infty}^{s_1} \left[k\, m_{po} - (k\, m_{po})^{-\left(\frac{s}{m_{po}}\right)^2} \right] PR(m_{po})\, dm_{po} \quad (10\text{-}29)$$

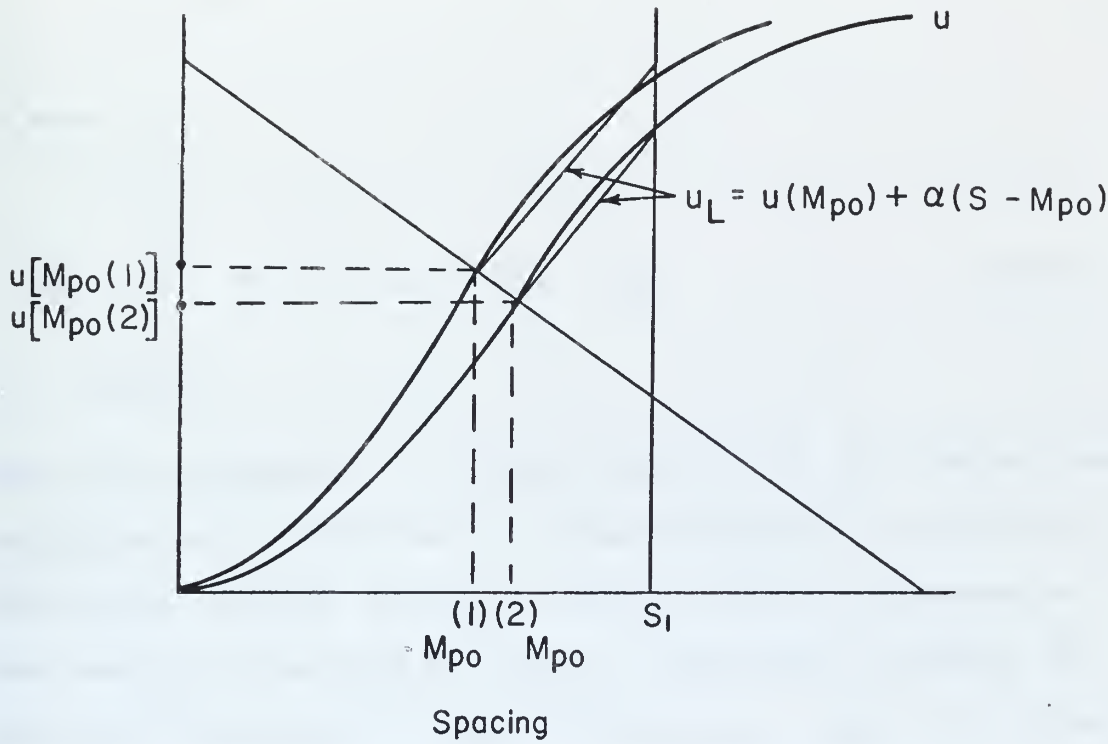
respectively.

Linear Approximation

Let us return to the integral I_2 to examine the results if we simplify the analytical expression of $U(m_{po}, s_1) - U(m_{po})$. First, let us note that the only portion of a particular undetected penetration curve that we are interested in, is that which corresponds to the range of spacing between m_{po} and s_1--that is, how U behaves between $U(m_{po})$ and $U(m_{po}, s_1)$. We will now assume that a linear function describes how any monotonic increasing U function varies between m_{po} and s_1. In particular we will assume that the linear function is of the following form for all members of the family:

$$U_L = U(m_{po}) + \alpha(s - m_{po}) \qquad m_{po} \leq s \leq s_1 \qquad (10\text{-}30)$$

where α is a known parameter, greater than zero, which we will speculate on later. This linear approximation is illustrated in Figure 24.



Linear Approximation of u

FIGURE 24

Substituting Equation (10-30) into the expression for I_2 yields

$$I_2 = \int_{m_{po}=-\infty}^{s_1} \left[\left[U(m_{po}) + \alpha (s_1 - m_{po}) \right] - U(m_{po}) \right] PR(m_{po})\, dm_{po} \tag{10-31}$$

or, simply that

$$I_2 = \int_{m_{po}=-\infty}^{s_1} \alpha (s_1 - m_{po})\, PR(m_{po})\, d\, m_{po} \tag{10-32}$$

which we can recognize as a left hand linear normal loss integral when the preposterior distribution is a normal distribution as in our case. Note that the quantity $\alpha(s_1 - m_{po})$ is greater than zero and equal to zero when m_{po} is less than and equal s_1, respectively. Further, observe that this quantity is monotonic increasing. Hence, no previous assumption about the undetected penetration function has been violated.

Using the previous discussion of I_1, we can readily represent I_2 as

$$I_2 = \alpha\, L_{LH}(s_1) \cdot SD(m_{po}) \tag{10-33}$$

And, as before, we have two cases to examine for $I_{LH}(s_1)$ depending on the relation between $E(m_{po})$ and s_1. For the case when $E(m_{po})$ is less than s_1, we have $L_{LH}(s_1)$ from the Linear Normal Loss Table using the

following expression for $L_{LH}(s_1)$:

$$L_{LH}(s_1) = L_N\left[Z_n = \frac{s_1 - E(m_{po})}{SD(m_{po})} \right] + Z\ s_1 \qquad E(m_{po}) \leq s_1$$

(10-34)

When $E(m_{po})$ is greater than the present spacing, we have

$$L_{LH}(s_1) = L_N\left[Z_{s_1} = \frac{E(m_{po}) - s_1}{SD(m_{po})} \right] E(m_{po}) \geq s_1 \qquad (10\text{-}35)$$

We can now state for the same reasons present in our discussion of I_1, that I_2 is an increasing function of the preposterior variance Vm_{po}. This agrees with the heuristically developed result pertaining to the general expression for I_2. We now have a simpler expression with which to work and fortunately, it can readily be combined with the expressions for I_1 to yield the following simpler expression for the expected value of sample information:

$$EVSI = \left[k\ L_{RH}(s_1) + \alpha\ L_{LH}(s_1) \right] SD(m_{po}) \qquad (10\text{-}36)$$

Even though the linear approximation leads to this simpler and most likely, more usable or practical expression for EVSI, a question that arises is how accurate is the linear approximation? We will forego a numerical investigation for answering this question because of the numerous possibilities which can arise, depending on such variables as the number of screening units, present spacing, line efficiency function, and other factors which also can take on a large number of

values. However, for the linear approximation to be accurate, it must depend a great deal on how the slope of the linear approximation is chosen.

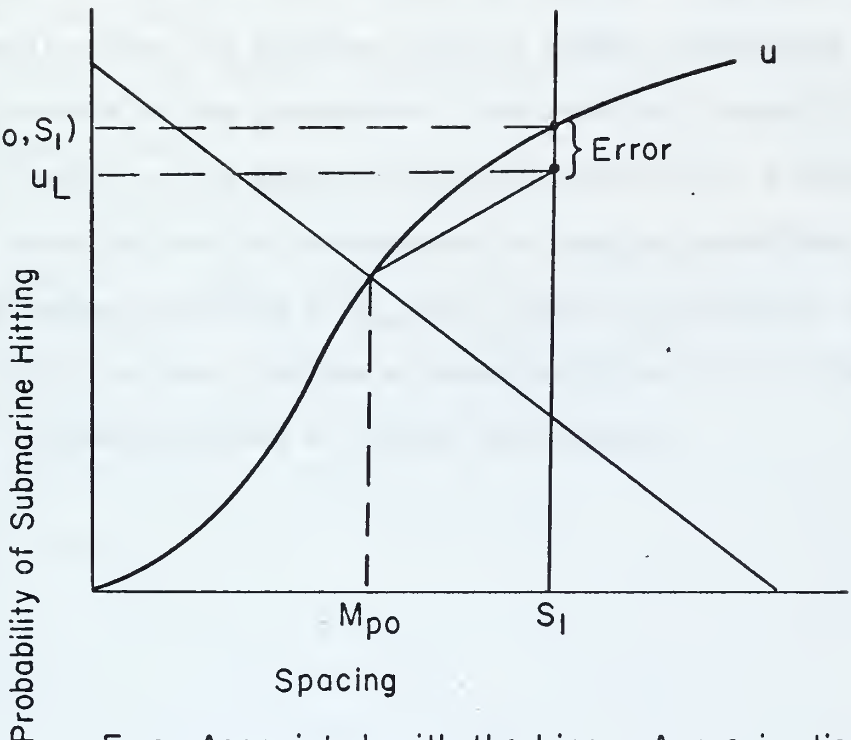
Using Figure 25, we can identify any inaccuracy associated with the linear approximation by the difference of the undetected penetration function for a given m_{po}, evaluated at the present spacing s_1, and the value calculated for the same m_{po} and s_1 by using expression (10-30). Calling this inaccuracy the error of the linear approximation, we can denote this deviation by $e(\alpha, m_{po})$ since it is a function of both the parameter α and the mean of the posterior distribution of optimal spacing. Thus, we have

$$e(\alpha, m_{po}) = U(m_{po}, s_1) - U_L \qquad m_{po} \leq s_1 \qquad (10\text{-}37)$$

A reasonable criterion which seems to suggest itself is to choose for the value of α that value which will minimize the expected absolute error or deviation. In order to do this, we must only optimize with respect to the parameter α the following equation which is a general summary of the suggested criterion:

$$\min_{\alpha} \left\{ E(e) = \int_{m_{po}=-\infty}^{s_1} \left| e(\alpha, m_{po}) \right| PR(m_{po}) dm_{po} \right\} \qquad (10\text{-}38)$$

Even though it may be more advisable to minimize the expected value of another function of the error, the criterion should be stated in order to identify explicitly the motivation for any method for choosing α which then follows.



Error Associated with the Linear Approximation

FIGURE 25

A Concise Statement

Let us now briefly report the findings of our operational analysis of the solution indicated by our model to the problem of when to sample BT information. We have found both for the general undetected penetration function and also the special case resulting from a linear approximation that the expected value of sample information increases as the variance of the preposterior distribution of spacing increases. Thus, it seems that we should be able to identify for a particular decision, since the cost of information is readily determined, what value of the standard deviation of m_{po} will result in a positive value of ENGSI. This, in turn, implies a simple decision rule in terms of $SD(m_{po})$ to determine when to obtain information.

CHAPTER XI

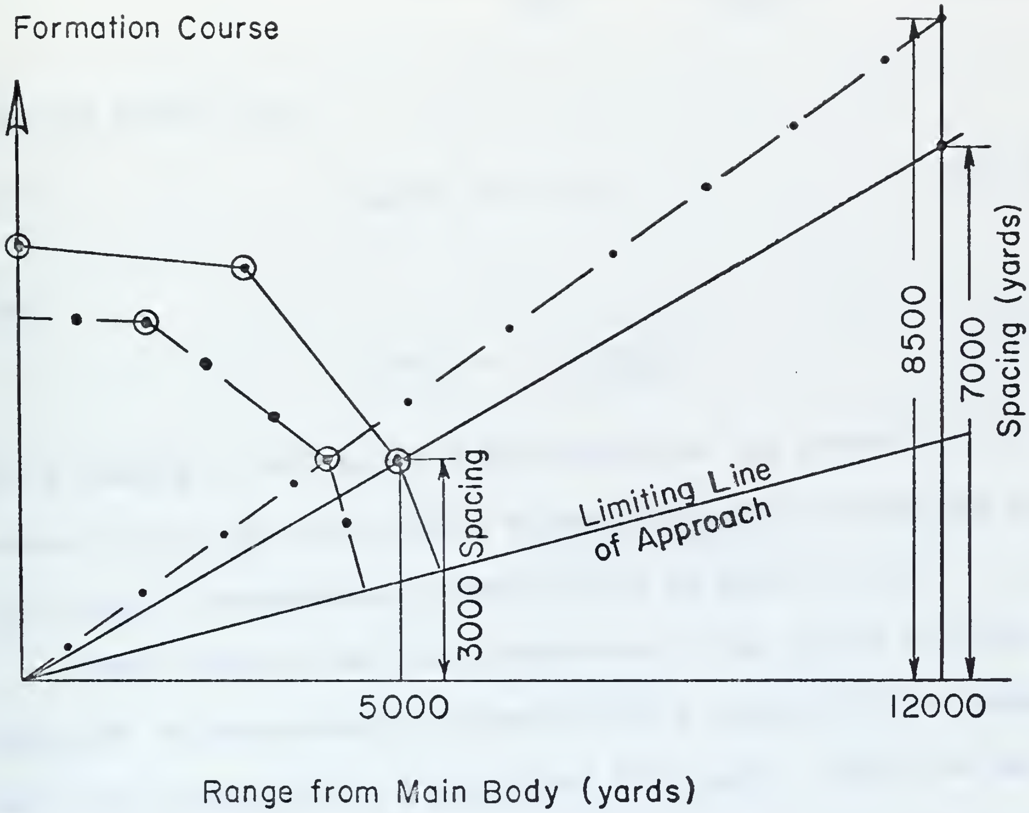
NUMERICAL EXAMPLE

Setting

We will now illustrate the use of our model of the screen placement problem by presenting a numerical example. Though the values used will be fictitious, they may be considered realistic enough so as to yield some feeling about the actual range of values for some of the parameters and variables used in the model.

Suppose we start out with a screen made up of five destroyers that are protecting a fleet oiler. Let us assume that the relative speed between the oiler and attacking submarine is estimated so as to yield limiting lines of approach that enclose an angle of 75 degrees, either direction from the formation course.

While a BT drop is being conducted, there will be only four destroyers in the protective screen. We may use Figure 26 to approximate the relationship between the distance from the oiler and the screen spacing for the given number of ships and assumed limiting lines of approach. Note that for a range of 5000 yards from the oiler to a destroyer, we estimate the spacing between destroyers to be 3000 yards. Suppose that the submarine's torpedo capability is estimated so as to yield some very small probability of submarine hitting at 12,000 yards or 6 nautical miles. This range would correspond to a spacing of 7000 yards. Hence, we let the slope, k, of our linear isoprobability



Approximation of Range - Spacing Relationship

— • — • — B T Drop Being Conducted

FIGURE 26

function be as follows:

$$k = \frac{1}{7000} \qquad (\frac{1}{\text{yards}}) \qquad (11\text{-}1)$$

Thus, we obtain from

$$p(s) = 1 - ks \qquad (11\text{-}2)$$

that

$$p(s) = 1 - \frac{1}{7000} s \qquad (11\text{-}3)$$

For a spacing of 3000 yards we are estimating the probability of submarine hitting the oiler with a torpedo fired just outside the screen at a range of approximately 4000 yards to be equal to .57.

When there are only four destroyers in the screen we again use Figure 26 to heuristically determine that a range of 12,000 yards from the oiler corresponds to a spacing of 8500 yards. Hence, we have

$$k_2 = \frac{1}{8500} \qquad (\frac{1}{\text{yards}}) \qquad (11\text{-}4)$$

For a spacing of 3200 yards while the BT drop is being conducted, the probability of submarine hitting with a torpedo is approximately equal to .63.

Let us assume that our present spacing s_1 is 3000 yards and that when a BT drop is made, the spacing s_2 will be 3200 yards. The cost of information can be found using the following expression

$$C_I = ks_1 - k_2 s_2 \qquad (11\text{-}5)$$

This yields

$$c_I = .43 - .37 = .06 \qquad (11\text{-}6)$$

In Figure 27 we have sketched an undetected penetration function and evaluated this function at several points. We have also included the linear isoprobability function. Note that the intersection of the functions, U and p(s), suggests that the values assumed for s_1 and s_2 to be reasonable as the screen commander attempts to follow the optimal screening rule. Because we will use the linear approximation we will extimate α by the following ratio:

$$\alpha = \frac{P(4000) - P(2000)}{4000 - 2000} \left(\frac{1}{\text{yards}}\right) \qquad (11\text{-}7)$$

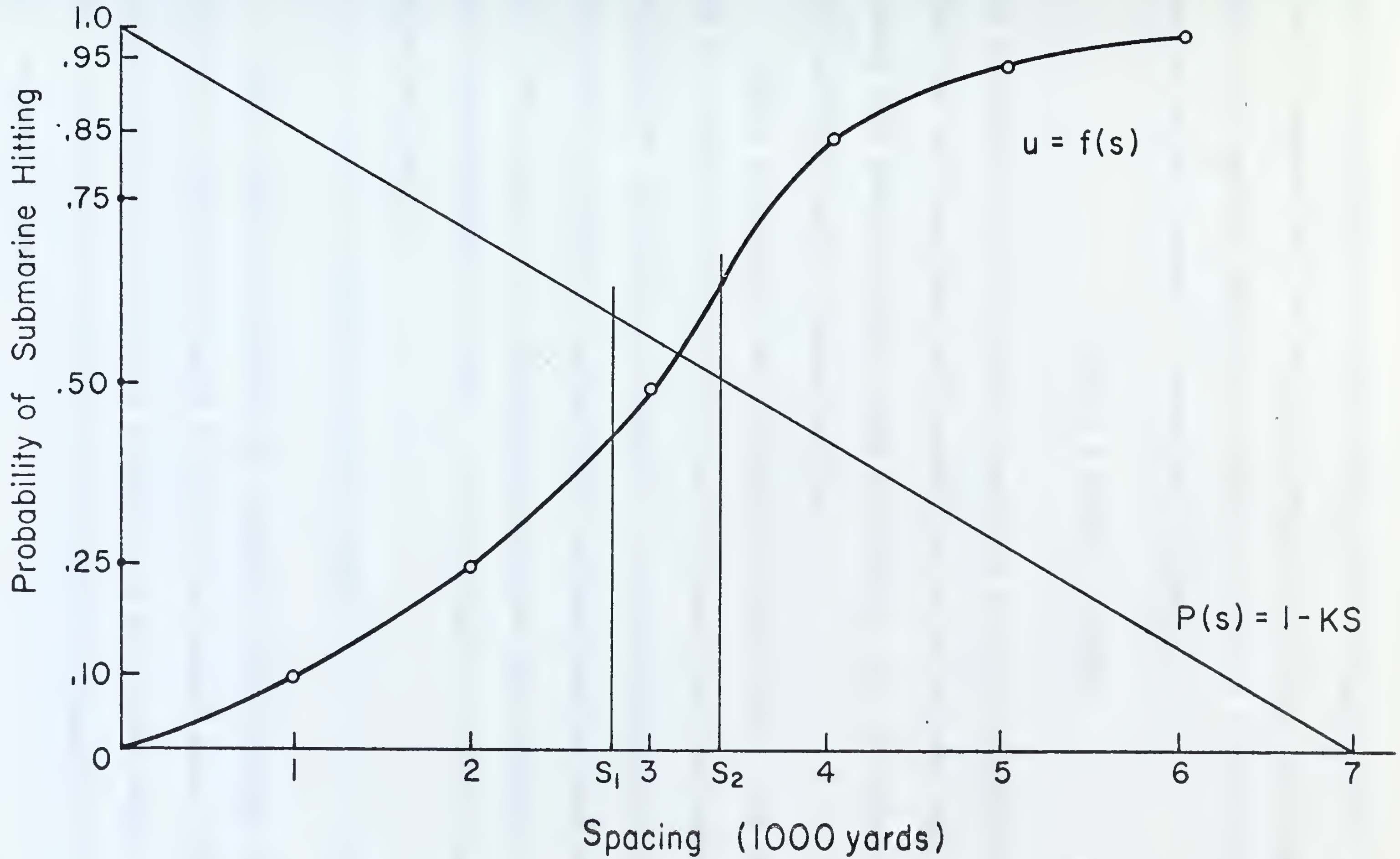
or,

$$\alpha = \frac{.85 - .25}{2000} = \frac{3}{10{,}000} \qquad (11\text{-}8)$$

Note that α is approximately twice the value of k.

ENGSI

Now, let us suppose that the screen commander feels that the water conditions have changed so that the effectiveness of his sonars have decreased. We will summarize his feelings by his prior distribution of optimal spacing which, in this case will indicate that the most likely value of optimal spacing will be greater than the present spacing since a deterioration of water conditions implies a modified undetected penetration function which will lie below the old U function.



ILLUSTRATING P(s)

FIGURE 27

Using the assumption of normal distribution throughout this example, we will suppose that the screen commander estimates that his prior mean of the optimal spacing is 3200 yards and his prior variance is equal to 10,000 (yards)^2. This can be summarized as

$$PR(s_o) = N(3200,\ 10{,}000) \qquad (11\text{-}9)$$

This assessment of the screen commander's personal probability indicates that he feels that the probability of the optimal spacing is between 3100 yards and 3300 yards is equal to .65, or between 3000 yards and 3400 yards is equal to .95.

Next, we consider the likelihood distribution. Let us assume that the screen commander feels that the procedure for obtaining and evaluating BT information is such that the sample spacing that results will be within 100 yards of the optimal spacing with a probability of .95. Hence, the standard deviation of the likelihood distribution is approximately 50 yards, or the variance V is 2500 (yards)^2. Thus we may write

$$LK(m_s \mid s_o) = N(s_o,\ 2500) \qquad (11\text{-}10)$$

With these distributions to summarize the screening situation, the screen commander now wants to know if he should make a BT drop. Hence, we need to evaluate the expected net gain from sample information. Our first step will be to formulate the preposterior distribution. Because we have

$$E(m_{po}) = m_{pr} \qquad (11\text{-}11)$$

we need only to find the variance Vm_{po} of the preposterior distribution. We have shown this variance can be calculated from the following expression:

$$Vm_{po} = \frac{V_{pr}^{2}}{V + V_{pr}} \tag{11-12}$$

Hence, we have

$$Vm_{po} = \frac{10^8}{2500 + 10^4} = 8000 \text{ (yards)}^2 \tag{11-13}$$

or that

$$SD(m_{po}) \simeq 90 \text{ yards} \tag{11-14}$$

Hence, before BT information is obtained, the screen commander's feeling about the resultant spacing if he goes ahead and conducts the BT drop is such that the best posterior act will be within 60 yards of the optimal spacing with a probability equal to .95.

We now want to calculate the expected value of sample information using the linear approximation for the undetected penetration function. We have already stated that $\alpha \simeq 2k$, so we can write

$$EVSI = k \left[L_{RH}(s_o) + 2L_{LH}(s_o) \right] SD(m_{po}) \tag{11-15}$$

Since m_{pr} is greater than s_1, we have

$$L_{RH}(s_1) = L_N \left[Z_{s_1} = \frac{E(m_{po}) - s_1}{SD(m_{po})} \right] + Z_{s_1} \tag{11-16}$$

or

$$L_{RH}(s_1) = L_N\left[\frac{3200 - 3000}{90}\right] + \frac{3200 - 3000}{90} \qquad (11\text{-}17)$$

$$L_{RH}(s_1) = L_N\,(2.2) + 2.2. \qquad (11\text{-}18)$$

In this case,

$$L_{LH}(s_1) = L_N\left[\,z_{s_1} = \frac{E(m_{po}) - s_1}{SD(m_{po})}\right] \qquad (11\text{-}19)$$

or,

$$L_{LH}(s_1) = L_N(2.2)\,. \qquad (11\text{-}20)$$

Because the quantity $L_N(2.2)$ is approximately zero, we will ignore it and write the following:

$$EVSI \simeq k\,(2)\,(2.2) \quad SD(m_{po}) \qquad (11\text{-}21)$$

$$EVSI \simeq \frac{1}{7000}\,(2)\,(2.2)\,(90) \qquad (11\text{-}22)$$

$$EVSI \simeq .056 \qquad (11\text{-}23)$$

$$EVSI \simeq .06 \qquad (11\text{-}24)$$

We now can write the expected net gain from sample information as follows:

$$ENGSI = EVSI - C_I \qquad (11\text{-}25)$$

or,

$$ENGSI \simeq .06 - .06 = 0 \qquad (11\text{-}26)$$

Thus, we obtain the result that the screen commander anticipates that additional BT information will not be beneficial so he will delay obtaining this information.

Suppose now that the screen commander was more uncertain than for the last result. That is, suppose the screen commander's prior variance was 14,400 (yards)^2. We now obtain that the preposterior variance is approximately 12,300 (yards)^2, which yields

$$SD(m_{po}) \simeq 110 \text{ yards} \tag{11-27}$$

Further, let us suppose that m_{pr} is equal to 3500 yards. In this case we can write expected value of sample information as follows:

$$EVSI \simeq \frac{1}{7000} (2) \left(\frac{500}{110}\right) (110) \tag{11-28}$$

or,

$$EVSI \simeq .14 \tag{11-29}$$

Thus, we have that the expected net gain from sample information is calculated to be as follows:

$$ENGSI \simeq .14 - .06 \tag{11-30}$$

$$ENGSI \simeq .08 \tag{11-31}$$

This calculation indicates that the screen commander should proceed to conduct a BT drop in order to obtain current information about the water conditions.

<u>Best Posterior Act</u>

In order to complete the numerical application of our model, let us suppose that the BT drop is conducted and the information is evaluated to yield a sample spacing value of 3200 yards. The

best posterior act or spacing ordered by the screen commander is calculated for m_{pr} equal to 3500 yards, as follows:

$$m_{po} = \frac{c\, m_{pr} + m_s}{c + 1} \qquad (11\text{-}32)$$

where

$$c = \frac{V}{V_{pr}} \simeq .25 \qquad (11\text{-}33)$$

Thus, we have

$$m_{po} \simeq \frac{.25(3500) + 3200}{1.25} \qquad (11\text{-}34)$$

or

$$m_{po} \simeq 3260 \text{ yards.} \qquad (11\text{-}35)$$

For this screening situation, the screen commander will order a screen spacing equal to 3260 yards. Further, we can summarize his revised estimate of the probability of submarine hitting that corresponds to his newly formed screen, or his posterior expected payoff, as follows:

$$\text{POEP} = 1 - k\, m_{po} \qquad (11\text{-}36)$$

or

$$\text{POEP} \simeq .54 \qquad (11\text{-}37)$$

So much now for illustrating the use of our Bayesian model of the screen placement problem. We turn our attention next to examining how sensitive is the Bayesian approach to the fundamental assumptions that were used to structure our decision problem.

CHAPTER XII

SENSITIVITY ANALYSIS

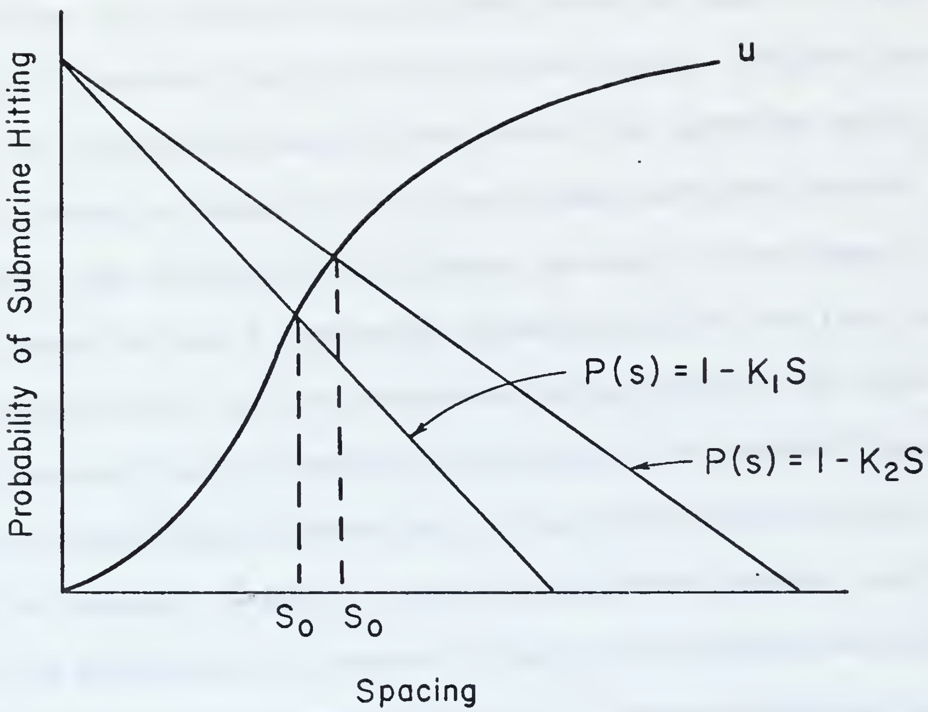
Preliminary Remarks

Our task to be accomplished in this chapter is to perform a sensitivity analysis in the broad sense. That is, we will investigate the Bayesian model of the screen placement problem in order to determine how dependent the Bayesian method of analysis is upon some of the significant assumptions that were used to structure the decision problem. Hopefully, the main assumptions can be changed without significantly altering the method of approach followed herein. Thus, by investigating changes or variations in, first the measure of effectiveness used to assign the payoff for the screen commander, and next, the decision variable or spacing used to identify the options or screens available to the screen commander, and finally, the screen commander's strategy that is used to choose the optimal screen, we intend to implicitly establish that the Bayesian approach can be viewed as a methodology for examining naval screen placement problems.

Measure of Effectiveness

We have elected to use as our measure of effectiveness, with which to assign values as the payoffs to the screen commander, the probability of submarine hitting the main body if, for a given screen, the submarine commander elects to either penetrate the screen or fire a single torpedo from just outside the screen. Determining and

revising the probability of submarine hitting that arises for the first option of the submarine commander, that which deals with penetrating the screen, is the main concern in the probabilistic and informational phases of our analysis. However, we have assumed as given the estimate of the threat and as part of this, the probability of submarine hitting the main body if he fires a single torpedo from just outside the screen. Suppose now we change this assumption which was used to structure the decision problem in the sense we will modify the capability of the submarine commander to launch torpedoes. Or, in other words, suppose that we are concerned with the probability of a submarine hitting the main body, if for a given screen, the submarine commander fires two torpedoes from just ouside the screen. The result from a change of this sort would be that the given estimate of the threat would show a greater probability of submarine hitting for all spacings, since we are now concerned with the probability of either of the torpedoes, or both, will hit the main body. Thus, if the slopes k_1 and k_2 are used to summarize the isoprobability contours ofprobability of submarine hitting for the original and changed assumption, respectively, then k_2 would be smaller than k_1, and if for the same undetected penetration function, U, the optimal screening rule is followed, then the optimal spacing would be greater for our changed assumption. This is illustrated in Figure 28. We are now in the position to point out that if any change in the given estimate of the threat will only modify the slope of the linear isoprobability function then the method of analysis undertaken herein will not be modified.



Revision of the Optimal Spacing
Due to a change in Slope of P(s)

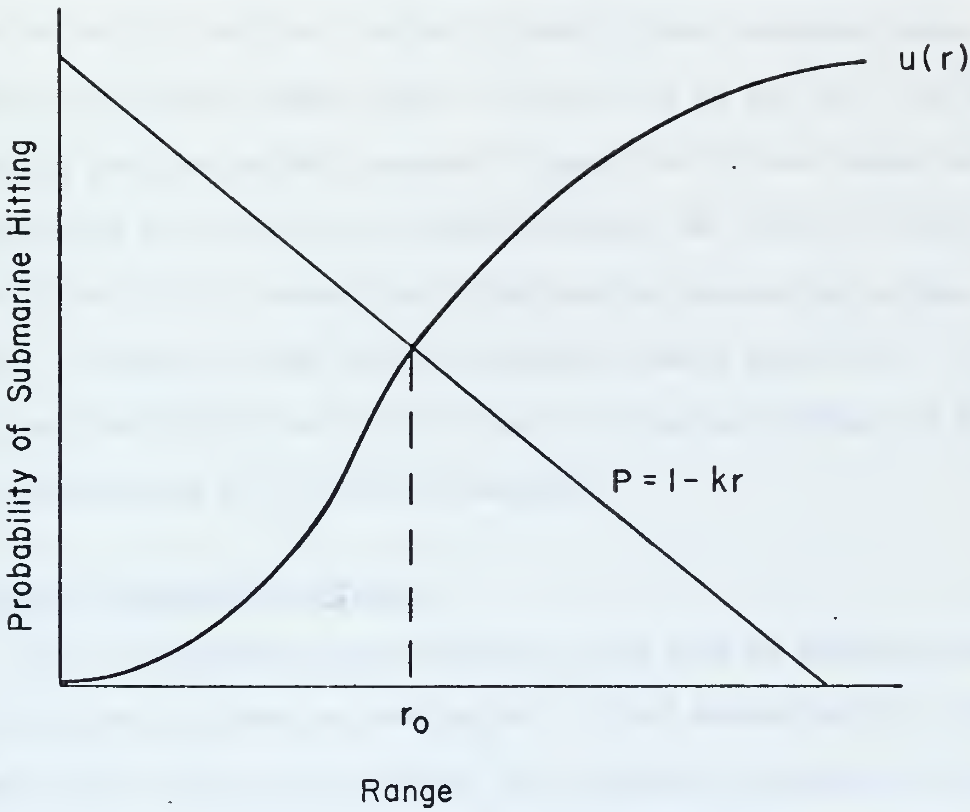
FIGURE 28

The Decision Variable

In Chapter V, we explicitly assumed the same lateral range curve for each screening unit. It then followed that for the same spacing between adjacent pairs of screening units we obtained the same probability of detection of a penetrating submarine and hence could summarize this probability associated with the complete screen by a measure called the line efficiency of the screen. Suppose, however, that we relax this assumption and permit the screening units to have various types of sonars or, in other words, different lateral range curves. The consequence of relaxing the same lateral range assumption would be that a complexity in calculation of the line efficiency of a screen but not the conceptual determination of the line efficiency because the same method of calculating the probability of penetration between each adjacent pair of screening units will not change in any manner. However, a complication arises because what is required is a combination of spacings between the available units that will lead not only to the same probability of detection between each pair, but also, the maximum probability of detection for the screening units in a particular type of screen at a given distance from the center of the main body. If we permit the above complication to enter our model, we can no longer use spacing as the controllable decision variable. However, the new decision variable will be the distance or range from the center of the main body at which to place the screen. Thus, we have not only line efficiency as a function of the distance or range, say r, but also, let us recall that we originally determined the

probability of submarine hitting with a torpedo as a function of the distance from the main body. Hence, this modification that leads to a new decision variable does not change the strategy by which the screen commander attempts to follow when making his choice but only modifies in the sense that the screen commander will desire to choose that range from the main body at which the probability of submarine hitting the main body if a torpedo is fired just outside the given screen, for this range is equal to the probability of detection of a submarine penetrating the same given screen for this range, given that a hit occurs for a successful penetration. Because the closer the ships are to the main body, the smaller the spacing between units will be, we can recognize that the modified undetected penetration function, say U(r), which now depends on the range from main body will have identical properties as the original undetected penetration function. That is, the closer to the main body the screen is formed, the smaller the probability of penetration. Further, we will assume that as the range from the main body at which the screen is placed is increased, the greater will be the probability of penetration. Hence, we can summarize this portion of the discussion of the change in the decision variable by Figure 29.

The next matter of concern is how our analysis will be modified if the decision variable, range from the main body, is used to structure the screen placement problem. A reasonable way to approach this modified problem is to assume that for a given mix of screening units, that the combination or arrangement of the units to provide a maximum,



The Optimal Screening Rule With the Decision Variable, Range From the Main Body.

FIGURE 29

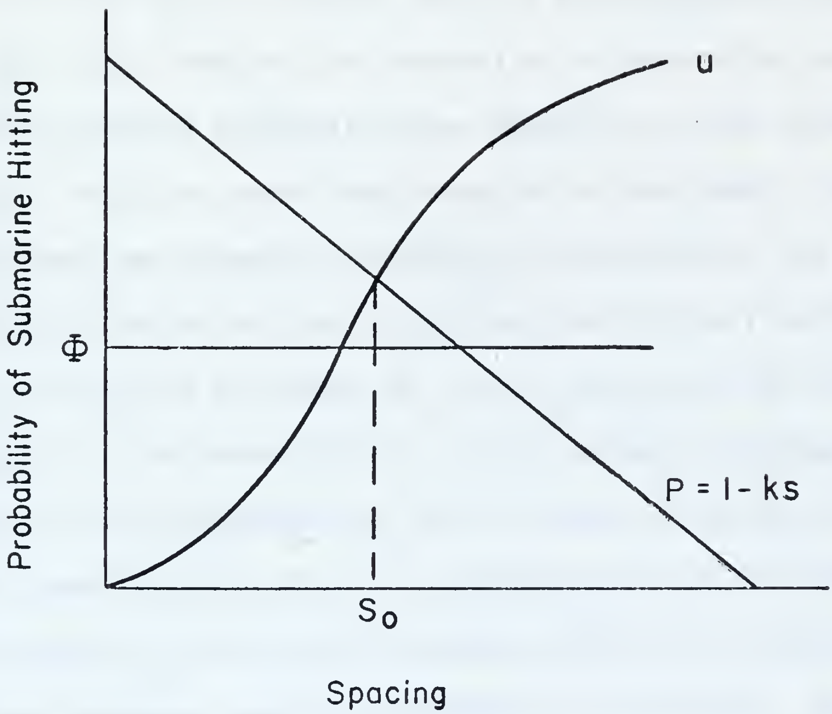
consistent line efficiency has already been determined and what decision is left for the screen commander is to determine the range from the main body to place his screening units in this given arrangement. Hence, we need only to modify the Bayesian approach to handle distributions that describe an optimal range, say r_0, that results from attempting to follow the modified optimal screening rule mentioned above. To concisely report the change that is required, is to say that only the guidelines that the screen commander attempts to follow change and corresponding to this are the quantities which we identify as the random variables of our probabilistic information processing system. Further, no change in the method of analysis seems appropriate in the sense that the PIP system is still used to order and revise the subjective probabilities of the screen commander.

The Screen Commander's Strategy

When we presented the assumptions that lead to applying the minimax criterion, in order to develop the optimal screening rule, we assumed that for any given spacing, the submarine commander will choose that option which will yield the larger probability of submarine hitting. Because we can view the screen commander's strategy as a function of the submarine commander's strategy, we will now change the above assumption, which was used to identify the submarine commander's strategy for the original model of the screen placement problem, in order to investigate if a Bayesian approach remains acceptable.

Let us assume now, that the submarine commander will not penetrate a screen with a given spacing, in order to attack the main body, unless the corresponding probability of submarine hitting is greater than some probability, say ϕ. Factors that may influence the reasonableness of the assumption are those, such as the safety of the submarine from attack, once the submarine is detected. That is, once the submarine commander attacks, from either inside or outside the screen, he is announcing the presence of a submarine, which will most likely lead to detection. Hence, we are now assuming that the submarine is more likely to survive if he is detected outside the screen than inside. However, we are not assuming that the screen commander will never attempt to penetrate the screen. We are modifying our assumption to indicate that the submarine commander will select that option which will yield the larger probability of submarine hitting, provided it is greater than some value ϕ when this larger probability corresponds to the option to penetrate. This change in the assumption of the strategy of the submarine commander yields two cases which must be considered. The cases are when the value ϕ is less than or greater than the probability which corresponds to the intersection of the linear isoprobability function and the monotonic increasing undetected penetration function.

We can summarize the first case when ϕ is less than the probability of submarine hitting that describes the intersection of the two previously mentioned functions by Figure 30. In this case, we can interpret the submarine commander's strategy as the same as that



Case One for a Modified Strategy
of the Submarine Commander

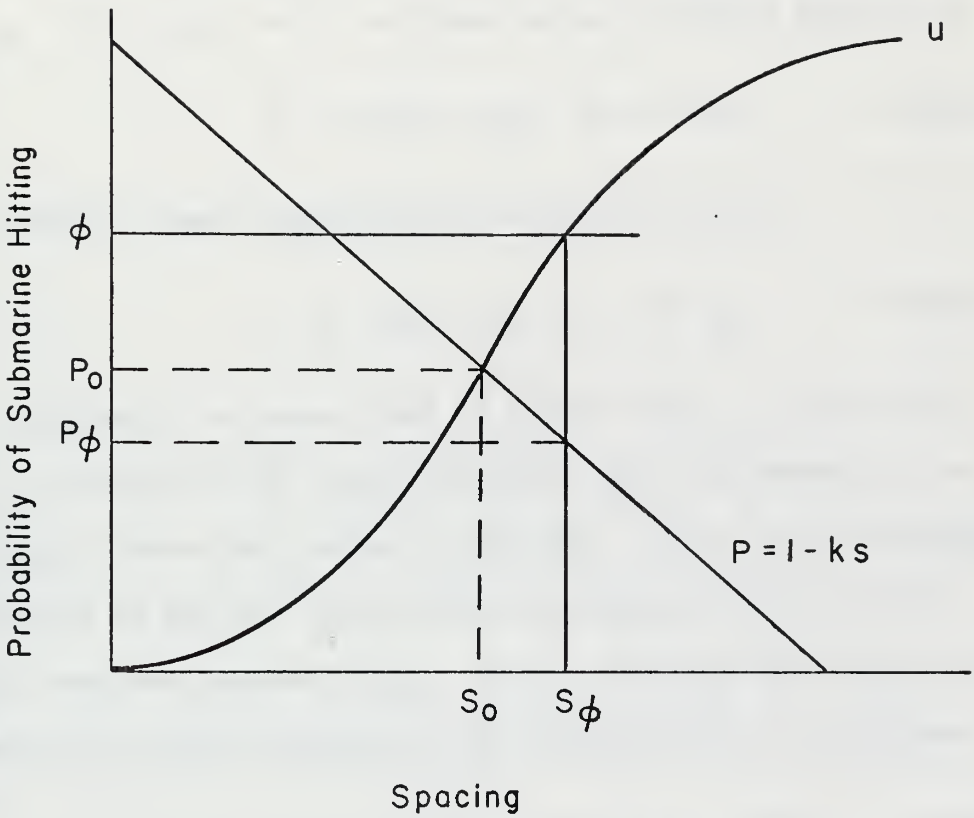
FIGURE 30

assumed to structure our original model of the screen placement problem. Hence, there will be no change in the strategy of the screen commander. Or, in other words, the optimal screening rule is applicable, which in turn, implies that no change in the Bayesian analysis of the decision problem of the screen commander is required.

Next, we need to examine the more interesting case when the value ϕ or lower limit on the probability of undetected penetration is greater than the probability that results from the optimal screening rule. This case seems more likely to be applicable, since we are assuming that the submarine commander can approximate the linear isoprobability function and the undetected penetration function. This case is illustrated by Figure 31. Let s_o and s_ϕ be the spacing that corresponds to the probability p_o if the optimal screening rule is followed and the probability p_ϕ which results if the screen is ordered with the spacing that yields an undetected penetration probability of ϕ. Note that s_ϕ is the largest spacing that can be used without the submarine commander selecting the option to penetrate. Hence p_ϕ is that probability of submarine hitting which corresponds to the submarine commander firing from just outside the screen with spacing s_ϕ. Hence, we have

$$p_\phi = 1 - ks_\phi \tag{12-1}$$

We can note now that if the spacing s_o had been ordered that the probability of submarine hitting for this decision would be given by



Case Two for a Modified Strategy of the Submarine Commander.

FIGURE 31

p_o, or, we have

$$p_o = 1 - ks_o \qquad (12\text{-}2)$$

Thus we see that the screen commander will incur a loss or increase in probability of submarine hitting if the ordered spacing s is less than s_ϕ. This loss, say L, is given by the following expression:

$$L = p(s) - p_\phi , \quad \text{if} \quad s \leq s_\phi \qquad (12\text{-}3)$$

or, using the linear isoprobability function, we have

$$L = k(s_\phi - s), \quad \text{if} \quad s \leq s_\phi \qquad (12\text{-}4)$$

When a spacing s is ordered that is greater than s_ϕ, a loss will also be incurred by the screen commander since the submarine commander will select the option to penetrate. Thus, the corresponding probability is not only greater than the value ϕ, but certainly greater than the probability p_ϕ. We can summarize the increase in probability in this situation by the expression for L as follows:

$$L = (1 - LE_{EST}\big|s) - P_\phi \qquad s \geq s_\phi \qquad (12\text{-}5)$$

where the quantity $LE_{EST}\big|s$ remains as before, as that value of the estimated line efficiency function evaluated at a given spacing s. We can simplify expression (12-5) to obtain

$$L = ks_\phi - LE_{EST}\big|s \qquad s \geq s_\phi \qquad (12\text{-}6)$$

Combining expressions (12-4)and (12-6), the following summary of the losses incurred by the screen commander if the screen that

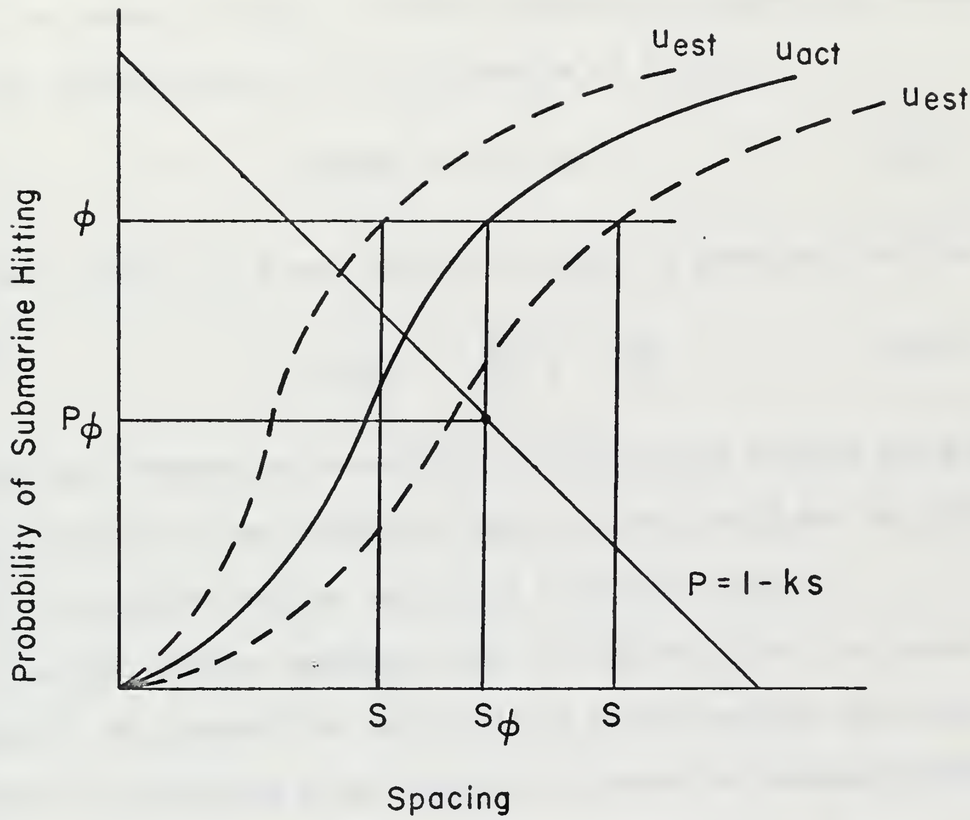
is ordered does not have the spacing s_ϕ:

$$L = \begin{cases} ks_\phi - LE_{EST}|s & s \geq s_\phi \\ k(s_\phi - s) & s \leq s_\phi \end{cases} \qquad (12\text{-}7)$$

But, this is the same expression for the screen commander's loss as we obtained before with the spacing s_o replaced by the spacing s_ϕ. Thus, if the screen commander ordered the spacing s_ϕ, his loss will be the minimum or zero. We will call the spacing s_ϕ the ϕ optimal spacing. Thus, for a given ϕ, the modified strategy of the screen commander will be to order the spacing whose corresponding undetected penetration probability is equal to the value ϕ. But, once again, because the line efficiency can change, the screen commander is uncertain about the ϕ optimal spacing just as he was uncertain about the value of the optimal spacing when he followed the optimal screening rule.

Using Figure 32, which is Figure 15 revised to reflect the second case of the modified strategy of the submarine commander, we can again observe that the smaller the deviation between the estimated line efficiency function and the actual line efficiency function, the smaller will be the loss for some ordered spacing that is not the ϕ optimal spacing. Note in Figure 32 that we are denoting the actual and estimated undetected penetration function by U_{ACT} and U_{EST}, respectively.

Let us simply summarize the change in our probabilistic information processing system by stating that the random process



Losses for Other Than ϕ Optimal Spacings

FIGURE 32

which the PIP system is designed to handle will become the uncertainty the screen commander has about his choice of spacing being the ϕ optimal spacing. Further, we can note that if the screen commander orders a spacing $\tilde{s}$, as he attempts to follow the ϕ optimal screen rule, we can, for reasons similar to those discussed in Chapter IX, express the prior expected payoff for this spacing as follows

$$\text{PREP}\big|\tilde{s} \;=\; 1 - k\tilde{s} \tag{12-8}$$

Thus, this leads to a ϕ gain function similar to Equation (9-12) as follows

$$g(s_\phi) \;=\; \left|k(s_\phi - \tilde{s})\right| \tag{12-9}$$

which we can interpret as describing the difference between the prior expected payoff for the particular choice of spacing $\tilde{s}$ and the probability of submarine hitting given the ϕ optimal spacing.

We have already indicated that the spacing s_ϕ is the random variable of the probability distributions which describe the screen commander's uncertaingy as he attempts to obtain the minimum probability of submarine hitting. Thus, when the uncertainty is described by a normal probability distribution as we have assumed the result or spacing to order is that spacing which corresponds to the expected value of the ϕ optimal spacing distribution. The prior and posterior best acts in this case are obvious from our previous developments of these best acts.

From Expression (12-7), which describes the screen commander's losses, it should be clear that an identical expression for the expected value of sample information results in the modified strategy, as for the strategy which followed from the optimal screening rule. Thus, we need only to show that the cost of information can be expressed by expression (9-28) to obtain that the expected net gain from sample information is provided by expression (9-43). Using Figure 33, where k_1 and k_2 represent the slope of the linear isoprobability function, before and after a screening unit is detached to obtain BT information, respectively, we can note that the spacing the screen commander orders will not change, but that the screen will be required to move closer to the main body in order to coincide with a larger isoprobability contour that represents the minimum probability of submarine hitting, while information is being sought. Thus, we can represent the cost of information as

$$C_I = p_2 - p_1 \qquad (12\text{-}10)$$

or,

$$C_I = k_1 s - k_2 s \quad . \qquad (12\text{-}11)$$

We are now in the position to describe analytically the expected net gain from sample information as follows.

$$ENGSI = EVSI - C_I \qquad (12\text{-}12)$$

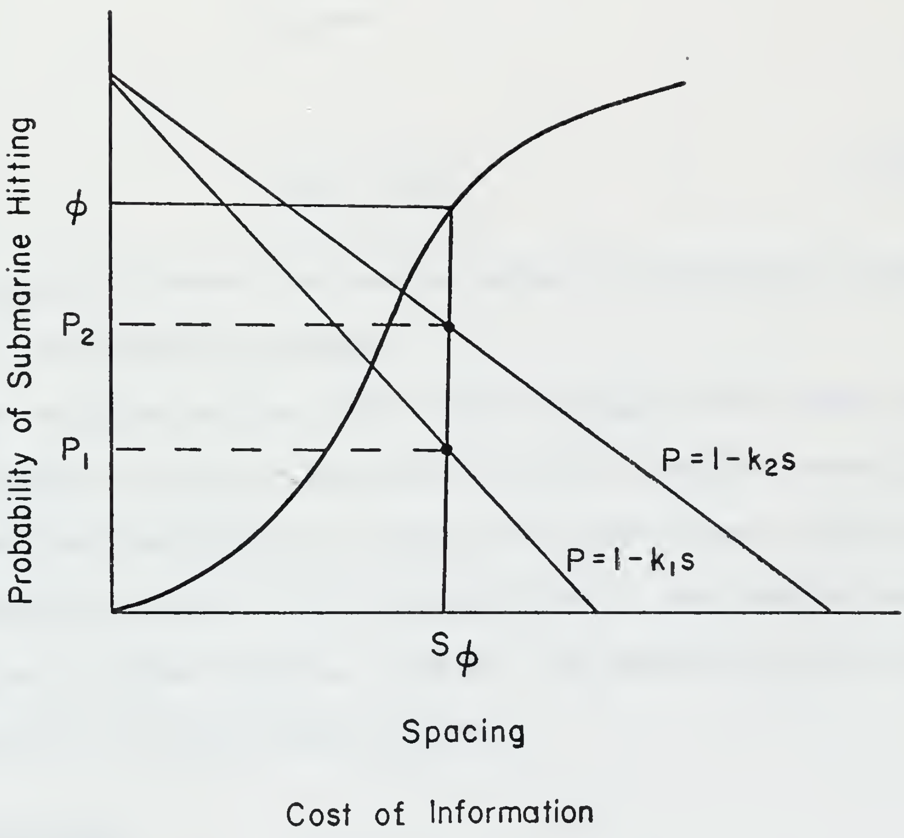

Probability of Submarine Hitting
ϕ
P_2
P_1
$P = 1 - k_2 s$
$P = 1 - k_1 s$
S_ϕ
Spacing
Cost of Information

FIGURE 33

or,

$$ENGSI = \int_{m_{po}=s_1}^{\infty} k(m_{po} - s_1) \; PR(m_{po}) \, dm_{po}$$

$$+ \int_{m_{po}=-\infty}^{s_1} \left[U(m_{po}, s_1) - (1 - km_{po}) \right] PR \; m_{po} \; dm_{po}$$

$$- (k_1 s_1 - k_2 s_1) \qquad (12\text{-}13)$$

where m_{po} now represents the expected value of the posterior distribution of the ø optimal spacing.

We can point out now that if the strategy of the screen commander changes, in that he attempts to follow a modified screen placement rule, then implicitly he modifies his thinking and explicitly he is required to express his uncertainty about some modified spacing, such as the ø optimal spacing. However, the Bayesian method of analysis has not in the least been modified.

A Consideration

Now that we have investigated changes in the significant assumptions that were used to structure the screen placement problem, and have shown no change is required in our method of approach for deriving a solution to this decision problem, we can state the

following consideration or claim:

> A Bayesian approach is, in general, a feasible technique with which to investigate screen placement problems.

CHAPTER XIII

COMMENTS AND EXTENSIONS

Preliminary Remarks

We must now examine the results of our theoretical analysis of the naval screen placement problem, in order to investigate the reasonableness of the interpretations that accompany these results. Or, in other words, we want to find out if our results should agree with the common sense of a screen commander. This type of investigation leads naturally to a comparison of the results of our model with current practice. Recommendations, research, and implementation are then discussed. Finally, our model is viewed as part of a decision problem which has time as a variable. A solution to this decision problem is suggested as an area of future research.

Conceptual Validations

We have already examined the probabilistic information processing system by means of its input and output probability distributions and found the interpretations that described the results of such a system to be reasonable. In these investigations, we used the variance of the probability distributions to indicate the reliability associated with expressions of the screen commander's uncertainty. We have now developed an expected net gain from sample information which is also influenced by the screen commander's reliability estimates. That is, we have an expression for the change in probability

of submarine hitting, that may be anticipated from sample BT information, which is a function of the variance of a particular probability distribution. Further, we have proposed best acts for the screen commander to make, either before or after information is obtained. We must now attempt to conceptually validate these results of our Bayesian analysis by showing that they lead to reasonable interpretations.

Recall that the expression for expected net gain from sample information is generally derived from the following equation

$$\text{ENGSI} = \text{EVSI} - C_I \tag{13-1}$$

In order to simplify our investigation, we will use the linear approximation, and write ENGSI as follows:

$$\text{ENGSI} = \left[K L_{RH}(s_1) + \alpha L_{LH}(s_1) \right] SD(m_{po}) - (ks_1 - k_2 s_2) \tag{13-2}$$

For the particular case when the present spacing s_1 is equal to the expected value of the preposterior mean, the prior mean of optimal spacing, both the left and right hand linear loss quantity is equal to .3989. We can then write

$$\text{ENGSI} = (K + \alpha)(.3989)\, SD(m_{po}) - (ks_1 - k_2 s_2) \tag{13-3}$$

From an examination of Equations (13-2) and (13-3), we observe that as the variance of the preposterior distribution increases

that ENGSI increases since

$$SD(m_{po}) = (V\ m_{po})^{\frac{1}{2}} \qquad (13\text{-}4)$$

That is, the smaller the amount of information, here the reciprocal of the variance, which the screen commander attributes to his prior distribution of the best posterior act, the more valuable he will consider sample BT information.

Because we have

$$Vm_{po} = \frac{V_{pr}^{2}}{V_{pr}+V} \qquad (13\text{-}5)$$

we can examine the effect of the prior variance of optimal spacing and the variance of the likelihood distribution on the screen commander's anticipated revision of probability of submarine hitting due to BT information.

From expression (13-5), we have that the variance of the preposterior distribution, Vm_{po}, increases as the prior variance of optimal spacing increases. Hence, the expected net gain from sample information increases as the prior variance increases. Or, in other words, as the screen commander becomes more uncertain about the optimal spacing before information is obtained, the more beneficial he anticipates sample BT information to be. Thus, as the reliability of the screen commander's estimates of the optimal spacing decreases, the more he realizes that any screen based on these estimates will most likely be in error. In this case, he realizes that BT information is more significant than if he held his estimates to be nearly certain.

Using expression (13-5) again, we can observe that as the variance of the likelihood distribution decreases, the greater the preposterior variance and hence, the greater the expected net gain from sample information. Thus, the more certain a screen commander is that the deviation between the sample value of spacing, indicated by his technique of processing the BT information, and the optimal value of spacing will be small, the more beneficial he will consider sample information to be. Or, we can say, the more reliable a screen commander feels his sample BT information will be processed to yield the optimal spacing, the more valuable sample information will be to him. This is certainly consistent with the result in Chapter VIII, that the more reliable a screen commander feels sample information to be, the more willing he is to let his decision be influenced by this information after it is obtained. Thus, we see that the screen commander's uncertainty about how BT information is evaluated to yield a proposed screen, influences not only the decision after the information is taken, but also, influences when a sample of BT information should be obtained.

From Equations (13-2) and (13-3), we can observe that as the value of the slope k, which is used to summarize the estimated threat, increases the expected value of sample information increases. This can be interpreted as indicating that for the same error in spacing, the threat with the greater value of k leads to the greater loss for the screen commander. Hence, the more information is worth. However,

even though the slope k_2 would become greater as k becomes greater, since they both describe the same weapon potential but only for different numbers of screening units, it is difficult to predict any net change in the cost of information as the value of k changes. Thus, we can make no general statement as to how the expected net gain from sample information changes as the value of k changes.

Turning our attention to the parameter α, or slope of the linear approximation, we can note that as the value of α increases, then the expected net gain from sample information increases. Or, we can interpret this as indicating that the greater the drop of the line efficiency of his screen as spacing increases, the greater will be the anticipated value of information. This seems reasonable because the greater the value of α the more will be the loss incurred for the same error in spacing, and hence, the more information will be worth to prevent this greater loss.

Examining expression (13-2) to investigate how ENGSI varies as the linear loss quantities $L_{RH}(s_1)$ and $L_{LH}(s_1)$ varies, we see that the relative values between the parameter k and α becomes significant. However, if k is approximately equal to α the value of the sum of $L_{RH}(s_1)$ and $L_{LH}(s_1)$, which are equal to .3989 for s_1 equal to $E(m_{po})$, will increase as the difference between the present spacing and prior mean of optimal spacing increases. Thus, the more the screen commander feels his present spacing is in error, the greater will be the anticipated benefits from sample information.

We can summarize these observations obtained from examining the expressions for ENGSI as validating our result for ENGSI in the sense that it leads to reasonable interpretations that may make sense to a screen commander.

If we examine now the best prior and posterior acts, which are indicated by our model of the screen placement problem, then we need to interpret the expected values of the prior and posterior distributions of the optimal spacing, respectively. The results of our model indicate that the screen commander should choose as his best acts, that value of spacing which he feels is most likely the optimal spacing, both before and after information is obtained. The simplicity of this decision rule is appealing and suggests that it would most likely make sense to a screen commander.

Comparison of Model Results with Current Practice

Our purpose in this section is to compare the results of our theoretical Bayesian analysis of the screen placement problem with the procedures that are currently practiced in the navy. The first result we need to examine is the one which states, for the case when a screen commander must decide on a screen before BT information is obtained, that he should order that spacing which he feels will mostly be the optimal spacing. We can only assume that when a screen is ordered before ships get under way from port, that an attempt is made to order a screen which most likely yields the minimum probability of of submarine hitting.

A second comparison to note is that the use of the PIP system to revise the prior probability distribution of optimal spacing, in light of additional BT information, incorporates the feelings or experience of the screen commander with the present doctrinal procedure of evaluating BT information. Thus, the present practice of evaluating BT information is not ignored, but its result, which has been referred to as the sample spacing, has been evaluated with regard to the reliability of this new information and then combined with the screen commander's feelings before the information was obtained. By doing this, we avoid the deterministic, aspiration filtering of information and maintain a measure of reliability that accompanies the results of our PIP system, upon which the screen commander's decision will be based. Or, in other words, we can supplement current practice in the navy with the experience or feelings of the screen commander.

Next, we must recall the current procedure of when to take a BT drop - once each watch, or more often, according to the screen commander's judgment. The results of our model indicate that the screen commander's judgment is important in deciding when to obtain BT information. This certainly agrees with current practice. However, our model has used the screen commander's judgment to quantitatively measure when the anticipated benefits from additional BT information is greater than the cost of obtaining this information. Hence, we now have a more systematic approach to yield when BT information can be expected to be of value to a screen commander.

This must be considered an improvement over the deterministic procedure of lowering the BT once every four hours or so. However, the Bayesian model of the screen placement problem has not identified in terms of time when a BT drop should be made. More will be said on this subject later.

Though we may view the results of our analysis to be more reasonable or more desirable to the screen commander, since they give him additional information with which to make his decision, a concern which we must now examine is the implementation of our model as the accepted procedure for the navy to use when deciding upon the placement of an ASW screen.

Recommendations

The following comments and recommendations are made to improve and support the ASW effort in the Navy. The main goal in this section is to imply the feasibility of the implementation of such investigations as conducted herein. Thus, some comments call for immediate action and others for a longer period before implementation can be achieved.

The role of education of a naval commander cannot be overemphasized. The education of a naval officer in the techniques of decision making and operations analysis, in order to at least provide him with a familiarity of the objectives and methods of these areas, should be established for officers. Thus, the recent requirement of establishing an undergraduate course in operations analysis in all Naval Reserve

Officer Training Corps programs is favorably viewed. To support this requirement, the textbook Naval Operations Analysis (22) has been revised to de-emphasize the mathematical techniques and published as Fundamentals of Naval Operations Analysis (9). It is recommended that all programs for new officers incorporate such a course to provide an introduction to the basic theoretic foundations upon which are based the tactics of the anti-air warfare, anti-submarine warfare, mine warfare, and other areas. It is also urged that the mathematical requirements be changed to include a basic course in probability theory. This recommendation will, hopefully, lead to more efficient communication and a better mutual understanding between a naval commander and the operations analyst who supports the commander through analytical efforts.

At the graduate level the continuation of the education of naval officers in special operations analysis programs is encouraged. Also, it is recommended that all Naval officers undertaking graduate education should be required to take a basic course in operations analysis. It is speculated that the implementation of these recommendations will eventually lead to a requirement for education in Bayesian Decision Theory. However, for now, it is recommended that all senior Naval officers be made aware of the benefits of Bayesian Decision Theory. Possibly this can be accomplished at the Naval War College level.

If recommendations such as those above are followed, then a naval commander may, some day, be expected to assess his uncertainty associated with problems, such as the naval screen placement problem, directly in the term of a probability distribution with its particular parameters. This would not only lead to an easier implementation of Bayesian analyses but also would permit the naval commander to immediately identify his role in such theoretical investigations.

We turn now, to recommendations that deal with the current research and data collection efforts presently underway. It is recommended that operational commanders periodically re-emphasize and revitalize the significance of programs such as the Fleet Antisubmarine Warfare Data Analysis Programs. It must be realized by each person involved in data collection during fleet exercises, that his role is critical in the overall analysis of a particular problem. After all, the output of investigations using information from fleet exercises is only as good as the input data.

In particular to the ASW problem, it is recommended that the use of the real time graphic display of an ASW problem, that is currently collected, in determining factors such as the sonar's lateral range curve for the particular BT conditions, be pointed out to the ASW team on board a ship. This may lead to better collective efforts.

It is also recommended that an investigation be conducted to compare the results that are described by the likelihood distribution. That is, first, the techniques for determining the optimal spacing

given a particular spacing is optimal should be evaluated as to their accuracy. Next, the screen commander's evaluation of this same likelihood distribution should be compared, in order to identify any bias or discrepancies. This may lead to interesting results such as the tendency of screen commanders to depend on the BT information, more than necessary; this would be the result if the variance of the likelihood distribution was continuously very small, even when error is possible. This type of result would immediately suggest studies that would identify to the screen commander how BT conditions may change as the surface water temperature. In turn an investigation into simplifying the judgmental process of estimating how the line efficiency changes as BT conditions change, might be made so as to identify significant decision variables, such as the time the surface temperature remains relatively constant.

Due to the current number of different sonars in the fleet, it is recommended that a study be conducted to determine the feasibility of changing the screen commander's decision variable from spacing to range from the main body. This would include a revision of screen placement tables in the sense that screens would most likely not remain symmetrical. Of course, reorientation of screens, difficulty in ordering a screen, and other such factors must be weighed before a change in the decision variable is considered.

An inquiry into the use of present shipboard computers to be programmed to accomplish the mathematics identified in this analysis

of the screen placement problem should be made. It seems reasonable that the AN/USQ-20 Computer that is used in the Naval Tactical Data System on board some of the newer ships in the fleet could be used to handle such calculations, since it is a high capacity, stored program, general purpose digital computer. These computers presently are used to accept and store information, update it, and provide it on request. This use seems to suggest that the programming of our PIP system is feasible on the AN/USQ-20.

It is recommended that the commanding officer or officer of the deck of a screening ship be advised, not only of the Effective Sonar Range, a particular lateral range that corresponds to a probability of detection equal to .5, but also, of the expected payoff or expected probability of submarine hitting upon which the current screen is based. This additional information may permit the visual lookouts to be particularly alerted for signs, such as the wake of a torpedo, or periscope of a submarine, depending on the screen placement rules that are being followed by the screen commander.

Following the recommendations that have been put forth as a result of the research undertaken to conduct this analysis of the screen placement problem, should lead to screening with smaller probability of submarine hitting. Also, other peripheral results may occur that will enhance the ASW capability of the Navy.

Future Extension

We have already pointed out that a question concerning real time could not be evaluated using the model developed herein. An analysis that can handle these types of problem may be considered an extension of this present treatment of the naval screen placement problem. The motivation of the extension of this study, in order to investigate problems dealing with real time, is furnished by the reasonable assumption that a screen commander's prior uncertainty about the optimal spacing may increase as time increases. That is to say, that the greater the time since the last BT drop, the greater the prior variance of the screen commander. This seems reasonable because the water conditions will have to change. Recall that we have shown that as the prior variance increases, the preposterior variance increases, which implies that the expected net gain from sample information increases. This in turn, suggests the interesting situation that even though the ENGSI is greater than zero, a screen commander may desire to wait awhile longer before conducting a BT drop since the value of ENGSI will increase. An interpretation of this suggestion is that the screen commander may be willing to accept the error due to the present spacing until some future time when he is willing to increase the probability of submarine hitting, when conducting a BT drop. An explanation of these actions of a screen commander which seems to suggest itself is that the screen commander may be influenced by the probability a submarine is present, which may change over time.

A proposal for future research into the above explanation that may be considered an extension of the research accomplished herein will now be outlined. This proposed research may be viewed as an extension since it uses the model of the screen placement problem to calculate and revise the probability of submarine hitting for a given screen. We shall now examine where our model fits.

Assuming that if a submarine is present it will attack, we will refer to the probability that a target is present as the probability of an attack. For some planning horizon, the determination of which may be interesting and difficult, a screen commander will indicate a probability distribution that describes the likelihood of an attack by a submarine. Here intelligence estimates and reports from other groups of ships may be important to the screen commander in establishing such a probability distribution. Also, the route the main body will be transversing will also be significant because if the water is relatively shallow, then a submarine commander may elect not to operate in such an area. When a screen commander feels an attack is equally likely over any period of time for some given planning horizon, say T, we may describe this situation by a uniform distribution, or we may write

$$P(A \text{ in } \Delta t) = K \qquad 0 \leq t \leq T \qquad (13\text{-}6)$$

where P(A) denotes probability of attack and t stands for time.

Using the probability of attack and the probability of submarine hitting that was determined in our Bayesian analysis, we may write the probability of survival, say P(SUR), of the main body in some increment of time, say Δt, as equal to the sum of the probability of no attack in Δt and the product of the probability a submarine does not hit, given an attack and the probability of attack in Δt. Or, symbolically, we may write

$$P(\text{Survival in } \Delta t) = 1 - P(A \text{ in } \Delta t) + P(A \text{ in } \Delta t)\left[1 - P(s)\right] \tag{13-7}$$

Using the uniform distribution for probability of attack we may write

$$P(\text{Survival in } \Delta t) = (1 - K) + K\left[1 - P(s)\right] \tag{13-8}$$

Assuming that the probability of attack is uniform for a given planning horizon T, we can describe how the model of the screen placement developed herein can be used to explain how the feelings of a screen commander about the probability of survival may change over time. Suppose we start out with some probability of survival, say $P(SUR_0)$. As time increases and without any additional BT information, the probability of survival will decrease since the screen commander may feel that the probability of submarine hitting is increasing because water conditions have changed without any adjustment in the screen. Hence, a screen commander is anticipating an error in present spacing which leads to a loss in payoff or increase in probability of submarine hitting. At some time, t_1, the ENGSI is evaluated as being

positive at which time he detaches a ship to obtain BT information. This can be modeled by a sharp decrease in probability of survival to reflect the cost of information. We may then assume that the probability of survival will continue to decrease as the information is being obtained. However, once the BT drop has been completed, say at t_2, we may model the probability of survival as immediately increasing, as the new screen is ordered in accordance with the best posterior act by the screen commander. This description is summarized in Figure 34.

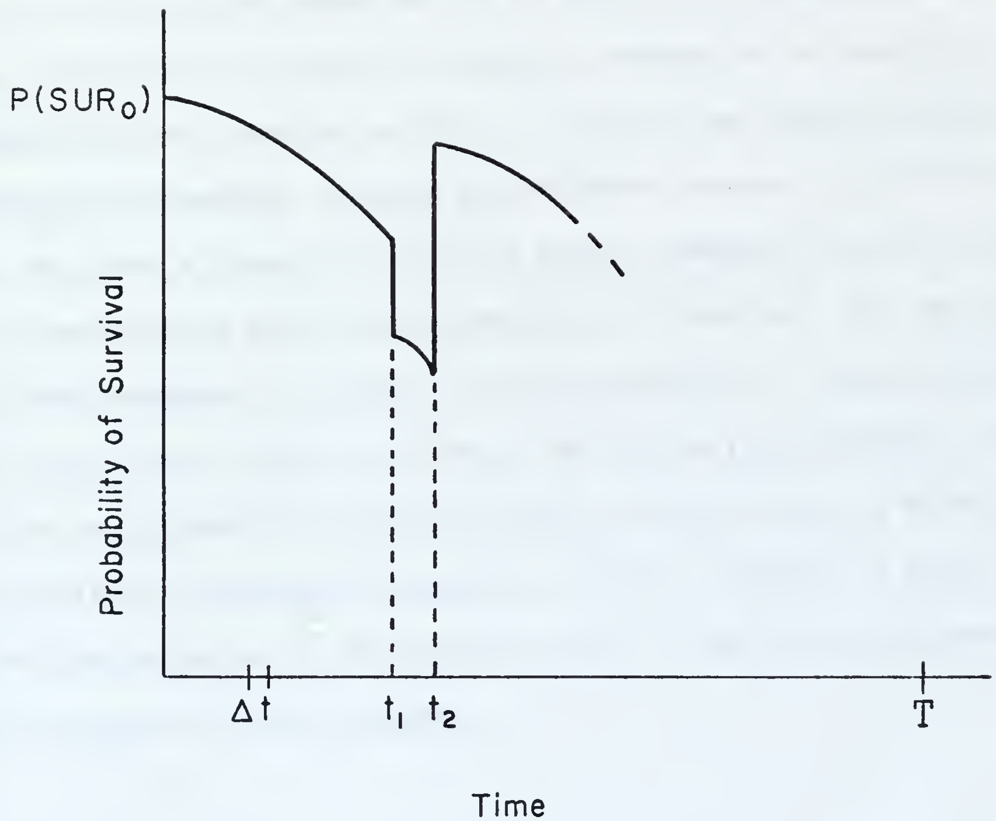
One further addition to this extension may be to look at the probability of survival over the complete planning horizon. Denoting this particular probability by P(T), we may write

$$P(T) = C_T \cdot \int_0^T P(\text{Survival in } \Delta t)\, dt \quad (13\text{-}9)$$

where C_T is a constant to insure that expression (13-9) conforms to the laws of probability theory. Let us note that if we represent P(T) as the ratio of the area under the curve in Figure 34 to the area of the rectangle with dimensions 1 x T, then we can write P(T) as follows:

$$P(T) = \frac{1}{T} \cdot \int_0^T P(\text{Survival in } \Delta t)\, dt \quad (13\text{-}10)$$

From this expression we can observe that as T gets very large



A Model of the Screen Commander's Personal Probability of Survival Versus Time.

FIGURE 34

(i.e., T approaches infinity) that P(T) approaches zero. This certainly seems reasonable because it indicates that a ship which may be continually attacked forever will be eventually hit.

Now that the nature of the proposed problem has been identified, many interesting questions can be asked. For example, what policy of BT drops is optimal so that the probability of survival is maximized for some planning horizon T? How can two proposed policies of BT drops be compared, or what is a suitable measure of effectiveness? How does a change over time in screen commander's variance of the optimal spacing affect the probability of survival? How should the screen commander's estimates of the probability of attack influence a BT policy? These are a few of the interesting questions that may need to be examined before the policy of when to take a BT drop as a function of time may be answered. Efforts extended in such a challenging extension of the Bayesian model of the screen placement problem should be quite rewarding.

CHAPTER XIV

SUMMARY AND CONCLUSIONS

We have applied Bayesian Decision Theory to investigate the protection of a main body of ships by screening ships against the attack of submarines. This investigation is anticipated to stimulate a more extensive theoretical investigation to establish a Bayesian methodology with which screen placement procedures can be analyzed and designed so as to be complementary.

The naval screen problem has been structured using kinematic search theory and the theory of games, and an optimal screen placement rule was determined. We have identified the loss of the screen commander which will result if his choice of spacing does not coincide with the optimal placement rule. This deterministic treatment of the screening problem has been interpreted through Bayesian Decision Theory to incorporate the experience or judgment of the screen commander. In particular, a probabilistic dimension was added to the decision problem through the use of a probabilistic information processing (PIP) system to revise the screen commander's personal probabilities in light of additional BT information. The role of the screen commander in assessing the probability distribution required by the PIP system was pointed out.

An informational dimension was added to the screen placement problem by an economic analysis which identified the cost and benefits

of additional information. The best course of action for the screen commander to take both before and after BT information was determined using the input and output of the PIP system. Reasoning, in a manner consistent with Bayesian Decision Theory, an analytical argument was presented to establish when the screen commander should obtain additional BT information.

An operational analysis was conducted and showed that the Bayesian model of the screen placement problem yielded plausible results. The effects of changes in factors, such as anticipated threat, detection capability of the screen, and uncertainty of the screen commander, on the solution to when to obtain BT information were investigated. We established that the more uncertain a screen commander is before information is obtained and the more reliable he considers the BT information to be, then the greater he anticipates the BT information is worth which implies the sooner he may desire to obtain BT information.

A sensitivity analysis was conducted to investigate the effect any change in a basic assumption, that was used to structure the decision problem, would have on the Bayesian Decision Theoretic approach. It was found that no significant change resulted and hence the feasibility of a Bayesian methodology for studying Naval screen placement problems was established.

Current doctrine that describes the frequency of BT drops that a screen commander should follow was compared with the theoretic

results. It was shown that the role of the screen commander's judgment in this problem could be quantified in order to assist the screen commander in making the decision when to conduct a BT drop.

The implementation of this Bayesian model was discussed. Both short and long-term recommendations were made to further the mutual understanding of a naval commander and operations analyst. Several interesting areas were identified to possibly assist in the implementation.

Finally, an extension of this analysis of the naval screen placement problem was outlined and suggested as future research. Several interesting questions were identified.

The goal of this analysis has been reached. Bayesian Decision Theory has been applied to the naval screen placement problem and yielded plausible explanations. This investigation provides now a possible methodology which may be used to evaluate current or proposed doctrine in this complex ASW area.

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On a Bayesian methodology to the solution of the Naval ASW screen placement problem.